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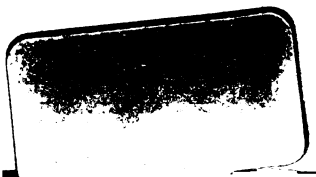

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PRACTICAL ALGEBRA

PRACTICAL ALGEBRA

(REVISED)

PREPARED FOR THE USE OF THE MIDSHIPMEN AT THE UNITED STATES NAVAL ACADEMY

BY

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THE UNITED STATES NAVAL INSTITUTE
1913

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PREFACE.

This revision of the Practical Algebra is designed as the first volume of a series of text-books written to meet the demand for a more condensed course in mathematics, necessitated by the increased time devoted to the professional subjects of marine and electrical engineering in the last two years at the Naval Academy.

To reach this end, the difficult algebraic proof of some theorems has been omitted, to be taken up later by the more simple and complete methods of the Calculus; and the time thus gained has been utilized for the solution of many problems designed to illustrate theory as well as to gain the important and necessary drill in numerical accuracy.

The methods of *Graphic Algebra* are carried through the analysis of linear and quadratic equations, with the end in view of eliminating the usual text-book on Conic Sections, which particular subject, together with Solid Geometry, will be given, as far as required by our course, in the introductory chapter of the Calculus.

We are indebted to Professor H. L. Rice, U. S. Navy, for valuable assistance.

U. S. NAVAL ACADEMY, August, 1910.

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CHAPTER I.

INTRODUCTORY.

1. Varieties of Numbers.—The numbers occurring in elementary mathematics are:

The *whole number*, also called *integer* or *natural number*.

The indicated imperfect quotient of two integers, or *common fraction*.

The indicated imperfect root of an integer, or *surd*.

The *negatives* of integers, fractions, and surds.

The indicated even root of a negative number, or an *imaginary*.

Integers and fractions are classed together as *rational* numbers, all others being *irrational*.

Besides imperfect roots, irrational numbers include such numbers as π, the ratio of the circumference of a circle to its diameter.

Rational and irrational numbers are classed together as *real* numbers. We can imagine all the real numbers arranged in magnitude on a scale, ranging from zero up through the positive numbers and down through the negative numbers; on this scale we can fix the position of a real number with any required degree of accuracy. Thus we can say that $\sqrt{6}$ lies between 2.44949 and 2.44948 because $(2.44949)^2$ is more than 6, and $(2.44948)^2$ is less than 6; it can be shown, too, that π is more than 3.14159 and less than 3.14160, and so on: any irrational number can be expressed approximately by a rational number, and so used in arithmetic processes.

Imaginary numbers, on the other hand, have no place in the scale of real numbers, but are of an entirely different kind.

They are useful because operations can be performed with them that lead to results concerning real numbers, and because in some branches of science, meaning may be attached to the imaginary number itself. When no meaning is attached to imaginary numbers their occurrence as answers to problems indicates impossibility of solution. This of course is equally true of real numbers; in a problem which asks for a number of men, such answers as $\frac{5}{3}$ or -1 are meaningless.

2. Operations with Surds.—There are two principles in accordance with which all simplification of surds must be performed:

I. Only integral multiples of the same root of the same number can be added or subtracted.

II. In multiplication or division,

$$\sqrt[n]{a}\,\sqrt[n]{b}=\sqrt[n]{ab};\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

For instance, $\sqrt{3}+\sqrt{6}$ cannot be simplified without evaluation, but

$$(3+\sqrt{6})-(5+\sqrt{54})=-2+\sqrt{6}-\sqrt{9}\sqrt{6}=-2-2\sqrt{6},$$

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{6}{9}}=\frac{\sqrt{6}}{3},$$

and so on.

3. Operations with Imaginaries.—The simplest imaginary is $\sqrt{-1}$, denoted by i. It is defined by the relation

$$\sqrt{-1}\sqrt{-1}=-1,\quad \text{or}\quad i^2=-1.$$

If a is a positive number, $\sqrt{-a}$ is an imaginary, defined by the relation

$$\sqrt{-a}=\sqrt{a}\sqrt{-1},\quad \text{or}\quad \sqrt{-a}=i\sqrt{a}.$$

Consequently,

$$\sqrt{-a}\sqrt{-b}=\sqrt{a}\sqrt{-1}\sqrt{b}\sqrt{-1}=\sqrt{ab}\cdot(-1)=-\sqrt{ab}.$$

It is important to note that this does not come under the law for the multiplication of surds; i. e., $\sqrt{-a}\sqrt{-b}$ is not equal to $\sqrt{(-a)(-b)}$, which would be $+\sqrt{ab}$, the symbol $\sqrt{}$ always indicating the positive square root. Until this distinction becomes familiar, the imaginary unit i should be kept separate in all operations.

For example, to multiply

$$(\sqrt{-2}+\sqrt{-3})(\sqrt{-2}-2\sqrt{-3}),$$

rewrite the factors:

$$(i\sqrt{2}+i\sqrt{3})(i\sqrt{2}-2i\sqrt{3}),$$

and multiply, remembering that $i^2=-1$:

$$\begin{array}{r} i\sqrt{2}+\ i\sqrt{3} \\ i\sqrt{2}-2i\sqrt{3} \\ \hline -2-\ \sqrt{6} \\ 6+\ 2\sqrt{6} \\ \hline 4+\ \sqrt{6}. \end{array} \quad \textit{Ans.}$$

In operations calling for higher powers of imaginaries, it is well to notice that

$$i^2=-1,\quad i^3=-i,\quad i^4=1,$$
$$i^5=i,\quad i^6=-1,\quad i^7=-i,\quad i^8=1,\quad \text{etc.},$$

the powers repeating in cycles of 4.

4. If $a+bi=c+di$, then $a=c$ and $b=d$.

For the equation may be written, $a-c=i(d-b)$; and since the difference between two real numbers can not, by the nature of their definitions, be equal to an imaginary number, this equation can be true only when $(a-c)=0$ and $(d-b)=0$, or $a=c$, $b=d$.

This same principle applies also to surds; thus, if

$$a+\sqrt{b}=c+\sqrt{d},$$

we should have $a=c$ and $b=d$, if only $\sqrt{b}$ and $\sqrt{d}$ are irrational numbers. For the equation may be written $(a-c)=(\sqrt{d}-\sqrt{b})$, in which form $(a-c)$ must be *rational* or *zero*, while $\sqrt{d}-\sqrt{b}$ can only be *irrational* or *zero;* hence $(a-c)$ and $\sqrt{d}-\sqrt{b}$ must each equal zero to satisfy the equation.

5. When two surds, or two imaginary expressions, differ only in the sign of the surd or imaginary part, such quantities are said to be *conjugate.*

Thus,

$$a-\sqrt{b} \text{ is conjugate to } a+\sqrt{b}$$

and

$$a-bi \text{ is conjugate to } a+bi.$$

6. The sum and product of two conjugate surds are rational; for

$$(a+\sqrt{b})+(a-\sqrt{b})=2a,$$
$$(a+\sqrt{b})\times(a-\sqrt{b})=a^2-b.$$

7. The sum and product of two conjugate imaginary expressions are real; for

$$(a+bi)+(a-bi)=2a,$$
$$(a+bi)\times(a-bi)=a^2+b^2.$$

8. Fractions involving surds in the denominator should generally be simplified, by *making the denominator rational* by multiplying both numerator and denominator by the conjugate of the denominator.

Thus,

$$\frac{1}{\sqrt{3}}=\frac{1}{3}\sqrt{3}, \quad \frac{a}{\sqrt{b}}=\frac{a\sqrt{b}}{b},$$

$$\frac{2-\sqrt{-3}}{3-\sqrt{-3}}\times\frac{3+\sqrt{-3}}{3+\sqrt{-3}}=\frac{9-\sqrt{-3}}{12}.$$

9. To Find the Square Root of $a+\sqrt{b}$.—Assume

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}.$$

Then

$$a+\sqrt{b}=x+y+2\sqrt{xy}$$

and (Art. 4)

$$x+y=a, \tag{1}$$

$$2\sqrt{xy}=\sqrt{b}. \tag{2}$$

Taking the difference of the squares of (1) and (2), and extracting the square root of the difference,

$$x-y=\sqrt{a^2-b}. \tag{3}$$

If $\sqrt{a^2-b}$ is rational, the value of $\sqrt{x}+\sqrt{y}$ is obtained from (1) and (3); if $\sqrt{a^2-b}$ is not rational, the expression $\sqrt{a+\sqrt{b}}$ can not be simplified.

10. To Find the Square Root of $a+bi$.—Assume

$$\sqrt{a+bi}=x+yi.$$

Squaring,

$$a+bi=x^2-y^2+2i\cdot xy.$$

Equating the real and imaginary parts,

$$x^2-y^2=a, \tag{1}$$

$$2xy=b. \tag{2}$$

Adding the squares of (1) and (2) and extracting the square root of the sum,

$$x^2+y^2=\sqrt{a^2+b^2}. \tag{3}$$

From (1) and (3) the values of x^2 and y^2 are easily obtained; and if $\sqrt{a^2+b^2}$ is rational, the expression may be simplified.

Example 1: Find the square root of $129-72\sqrt{3}$.

We have from (1) and (2),

$$x+y=129,$$
$$2\sqrt{xy}=72\sqrt{3},$$

whence

$$x-y=33,$$
$$x=81,$$
$$y=48,$$

and

$$\sqrt{x}-\sqrt{y}=9-4\sqrt{3}.$$

Example 2: Find the square root of $-47+8\sqrt{-3}$.

Assume

$$\sqrt{-47+8\sqrt{-3}}=x+y\sqrt{-1}.$$

Then

$$-47+8\sqrt{3}\cdot i=x^2-y^2+2xyi.$$

Equating the real and imaginary parts,

$$x^2-y^2=-47,$$
$$2xy=8\sqrt{3},$$

whence

$$x^2+y^2=49,$$
$$x^2=1,$$
$$y^2=48,$$

and

$$x+yi=1+4\sqrt{-3}.$$

Examples.

1. Rationalize the denominators of

$$\frac{25\sqrt{3}-4\sqrt{2}}{7\sqrt{3}-5\sqrt{2}},\quad \frac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}},\quad \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}.$$

2. Find the square roots of the following surds:

$$172-96\sqrt{3},\ 454+126\sqrt{5},\ 92-8\sqrt{60}.$$

3. Multiply $2\sqrt{-3}+3\sqrt{-2}$ by $4\sqrt{-3}-5\sqrt{-2}$.

4. Multiply $x-\frac{1+\sqrt{-3}}{2}$ by $x-\frac{1-\sqrt{-3}}{2}$.

5. Rationalize the denominators of

$$\frac{1}{2-\sqrt{-3}}, \quad \frac{3\sqrt{-2}-2\sqrt{-3}}{3\sqrt{-2}+2\sqrt{-3}} \text{ and } \frac{1-\sqrt{-3}}{1+\sqrt{-3}}.$$

6. Simplify $\frac{x+i}{x-i}-\frac{x-i}{x+i}$ and $\frac{a+ix}{a-ix}-\frac{a-ix}{a+ix}$.

7. Find the square roots of the following imaginaries:

$$5-12\sqrt{-1}, \quad 47-8\sqrt{-3}, \quad -8\sqrt{-1}.$$

8. Find in their simplest forms the roots of

$$x^4-196x^2+4=0 \quad \text{and} \quad x^4+150x^2+4761=0.$$

Ans. $\pm(5\sqrt{2}\pm4\sqrt{3})$, $\pm(\sqrt{-3}\pm6\sqrt{-2})$.

11. Quadratic Equations.—Any quadratic equation in one unknown quantity, x, can be written

$$ax^2+bx+c=0,$$

where a, b, and c are known numbers. Every such equation is satisfied by either of the two values

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x_2=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

the two *roots*.

It is possible to determine some facts about the roots of any quadratic equation without actual solution.

If b^2-4ac is less than zero (written $b^2-4ac<0$), the roots are two conjugate imaginaries.

If b^2-4ac is greater than zero (written $b^2-4ac>0$), the roots are real.

If $b^2-4ac=0$, the roots are equal.

If $b^2-4ac>0$ and a perfect square, the roots are rational.

The last relation is of no great importance, but the others are very important.

The quantity b^2-4ac, since it discriminates among the possible sorts of numbers, is called the *discriminant* of the equation.

The sum and product of the roots of any quadratic equation can be found by immediate inspection of the equation, for

$$x_1+x_2=-\frac{b}{a},$$

$$x_1x_2=\left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{b^2-(b^2-4ac)}{4a^2},$$

$$x_1x_2=+\frac{c}{a}.$$

Thus, in the equation $5x^2-79.2x+3.28=0$,

$$x_1+x_2=15.84, \quad x_1x_2=.656;$$

the roots are evidently real.

12. Maxima and Minima.—The greatest value an expression can have is called its *maximum* value; the least, its *minimum* value. Some problems concerning maxima and minima can be solved by applying the principles of the preceding article.

Example 1: Given $y=3+5x-2x^2$, what is the maximum value y can have if x is real, and what is the value of x that gives this maximum value to y?

Find x in terms of y:

$$x=\tfrac{1}{4}(5\pm\sqrt{49-8y}).$$

Clearly the largest value possible for y, if x is to be real, is $\frac{49}{8}$; and when y has this value, $x=\frac{5}{4}$. If the value of x had not been required, it would have been sufficient to equate to zero the discriminant of

$$2x^2-5x-3+y=0,$$

getting

$$25-8(-3+y)=0,$$

whence

$$y=\tfrac{49}{8}.$$

The following examples are similar.

Example 2: What limitation is there on the value of $3x^2-10x+16$ if x is real?

Putting $3x^2-10x+16=y$, the discriminant is $12y-92$, which is positive only if y is greater than $\frac{23}{3}$. $\frac{23}{3}$ is the *minimum* value; y may be $\frac{23}{3}$ or anything greater.

Example 3: If $9y-9y^2=x^2+2$, and both x and y are real, what limitations are there on the values of x and y?

Solving for x,

$$x=\pm\sqrt{(3y-1)(2-3y)}.$$

Solving for y,

$$y=\pm\tfrac{1}{3}\sqrt{(\tfrac{1}{2}-x)(\tfrac{1}{2}+x)}+\tfrac{1}{2}.$$

From the first result, it is evident that if y is $<\frac{1}{3}$ or $>\frac{2}{3}$, x is imaginary; from the second result, if x is $<-\frac{1}{2}$ or $>\frac{1}{2}$, y is imaginary. y may have any value from $\frac{1}{3}$ to $\frac{2}{3}$, x any value from $-\frac{1}{2}$ to $\frac{1}{2}$.

Examples.

In each of the following equations decide without solution, and if possible without getting the exact value of the discriminant, what the nature of the roots is, and also give the sum and the product of the roots:

1. $3x^2-19x+4=0$.

   Ans. Roots are real; sum, $\frac{19}{3}$; product, $\frac{4}{3}$.

2. $5x^2+28x-65=0$.

   Ans. Roots real; sum, $-\frac{28}{5}$; product, -13.

3. $px^2+qx=r$, p and r being positive.

4. $px^2+qx+s=0$, p and s being positive.

5. $77x^2-139x+62=0$.

Find the limitations on the values of the following expressions if x must be real:

6. $7x-3-2x^2$ and $9x-14-x^2$.

   Ans. First part, $3\frac{1}{8}$ maximum value.

7. $x^2-2x-15$ and $2x^2-x-6$.

   Ans. First part, -16 minimum value.

8. $\frac{x^2-5}{3}$ and $\frac{7-x^2}{3}$.

Ans. First part, $-\frac{5}{3}$ minimum value.

9. For what value of x is $(x-a)(b-x)$ a maximum?

Ans. $\frac{a+b}{2}$.

Find the limitations imposed on x and y by the following relations, if x and y are each real:

10. $x^2+y^2=7y-12$. *Ans.* $y<4$ and >3, $x<\frac{1}{2}$ and $>-\frac{1}{2}$.

11. $x^2+y^2=x+6$. *Ans.* $x>-2$ and <3, $y<\frac{5}{2}$ and $>-\frac{5}{2}$.

12. $x^2-y^2+6y=5$. *Ans.* $y<1$ or >5, x unlimited.

13. $y^2+2x+8=x^2$. *Ans.* $x<-2$ or >4, y unlimited.

14. Find the limitations on x and the maximum value of the radical in $y=\frac{1}{3}(2x+1)\pm\sqrt{(x+3)(2-x)}$, if x and y are each real.

CHAPTER II.

COMPUTATION.

13. Contracted Multiplication.—The computations that occur in the applications of mathematics generally deal with numbers that are not exact; consequently they are designed to attain only the degree of accuracy needed, or consistent with the approximate data on which they are based.

If, for instance, the sides of a rectangle are measured to the nearest hundredth of an inch and found to be 33.24 in. and 18.17 in., the most that we can feel sure of is that one side is between 33.235 in. and 33.245 in., the other between 18.165 in. and 18.175 in.

The area of the rectangle is found by the following multiplication, where the six asterisks indicate the absence of figures the value of which is unknown, and the last three figures of the product are clearly inaccurate. The result, read to the four significant figures which can be trusted, is 604.0.

33.24	
18.17	
2.3	268
3.3	24*
265.9	2**
332.4	***
603.9	708
604.0	

In contracted multiplication the process of multiplication is reversed, beginning with the highest digit of the multiplier, and carrying each partial product only as far as the figures are reliable. We thus begin with the left-hand figure of the multiplier, and place the result so that the decimal point falls underneath that of the multiplier.

33.24	
18.17	
332.4	*
265.9	2
3.3	2
2.3	3
604.0	

In multiplying by the successive digits, we cut off one figure of the product at each step, but *carry the tens* of the rejected figure to give the first one retained its proper value.

If we had made use of the values between which the true lengths lie, we should have found that the area lies between

$$33.235 \times 18.165 = 603.7$$

and

$$33.245 \times 18.175 = 604.2.$$

The area found by the contracted method, which is approximately the mean of the two limits between which the true area lies, is as close as the measurements warrant.

14. Contracted Division.—In the process of contracted division the useless figures of the dividend are cut off in a similar way; and we cut off a figure from the divisor at each step, carrying the nearest ten of the discarded figure into the partial product. Thus, if we wish to find the reciprocal of 2.302585 to the nearest seventh place of decimals, the work is performed as follows:

```
                0.4342945
              ------------
2.302585)1.
           .9210340
           ---------
2.30258ø)  .0789660
             690776
             ------
2.3025øø)    98884
             92103
             -----
2.302øøø)     6781
              4605
              ----
2.30ø2øøø)    2176
              2072
              ----
2.3øø2øøø)     104
               92
               --
2.øøø2øøø)      12
```

Examples.

In each of the examples, carry the work one place further than is required in the answer. Zeros between the decimal point and the first figure of a number, or zeros between the last reliable figure of a number and the decimal point, do not count as "significant figures."

1. Find the reciprocal of 0.4342945 to six significant figures.
Ans. 2.30259.

2. Find the square of 3.1416 to five significant figures.
Ans. 9.8696.

3. Find the square of 0.2135 to four decimal places.
Ans. 0.0456.

4. Find the square of 0.01723 to five decimal places.
Ans. 0.00030.

In each of the following, note what part of each fraction the next one is, and so simplify the computation, unless some obvious device is easier.

5. Find to five decimal places

$$1+\frac{x}{5}-2\left(\frac{x}{5}\right)^2+6\left(\frac{x}{5}\right)^3-21\left(\frac{x}{5}\right)^4+80\left(\frac{x}{5}\right)^5,$$

(a) when $x=0.1$, (b) when $x=0.12$.
Ans. 1.01924+ and 1.02292+.

6. Evaluate $4\left[4\left(\frac{1}{5}-\frac{1}{3\times5^3}+\frac{1}{5\times5^5}-\frac{1}{7\times5^7}+\frac{1}{9\times5^9}\right)-\left(\frac{1}{239}-\frac{1}{3\times239^3}\right)\right]$,

getting the result correct to four decimal places.

7. Find to five decimal places

$$1+\tfrac{1}{2}+\tfrac{1}{2}(\tfrac{1}{2})^2+\tfrac{1}{6}(\tfrac{1}{2})^3+\tfrac{1}{24}(\tfrac{1}{2})^4+\tfrac{1}{120}(\tfrac{1}{2})^5+\tfrac{1}{720}(\tfrac{1}{2})^6,$$

and square the result, getting the final answer to five decimal places. *Ans.* 2.71828.

8. Multiply 0.6931472 by 0.4342945 to seven decimals.
Ans. 0.3010300.

15. Logarithms.—A far greater saving of time and labor in the multiplication, division, involution and evolution of numbers is effected by the use of *logarithms*. The definition and use

of logarithms are based on the principles of exponents or indices, already known to the reader, which are summarized in the next article.

16. Laws of Indices.—If n is a positive whole number, a^n represents the product of n factors, each equal to a, and

$$a^n \cdot a^m = a^{n+m}, \tag{1}$$

$$(a^n)^m = (a^m)^n = a^{mn}, \tag{2}$$

$$\frac{a^n}{a^m} = a^{n-m}, \tag{3}$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}. \tag{4}$$

The above relations hold true whether any of the exponents involved are fractional or negative, or both.

From (3), if $n=m$, we have $a^0 = \frac{a^n}{a^n} = 1$, and when $n=0$, we have $\frac{1}{a^m} = a^{-m}$.

LOGARITHMS.

17. If we multiply together three 2's, the result is 8; this relation is expressed, according as we emphasize the result, the value of a single factor, or the number of factors, in three ways:

8 is the third power of 2:	$8 = 2^3$;
2 is the third root of 8:	$2 = \sqrt[3]{8}$;
3 is the logarithm of 8 to the base 2:	$3 = \log_2 8$.

18. Definition.—In the equation $b^l = n$, *the exponent* l is called *the logarithm of* n *to the base* b; thus, the logarithm of a number (n) to a base (b) is the power (l) to which the base must be raised to produce the given number. This relation is usually written

$$\log_b n = l.$$

The number n is frequently called the *antilogarithm* of l to base b, and may be written

$$n = \log_b^{-1} l,$$

where the symbol $\log^{-1}$ is read "antilogarithm."

Any positive number except unity may be taken as the base of a system of logarithms; negative numbers are excluded because the powers of negative numbers may be positive, negative, or imaginary, so that it is impossible to represent all numbers as powers of a negative number.

By a system of logarithms is meant the logarithms of all positive numbers to a given base. When it is understood that a particular system of logarithms is to be used, the suffix denoting the base is omitted, and we write, for example, log 2, instead of $\log_{10} 2$, where 10 is the base of the system.

19. Properties of Logarithms.—We have always $b^l = n$ and $\log_b n = l$ as equivalent expressions; that is, b to the power l equals n, and the logarithm of n to the base b equals l, mean the same thing. The properties of logarithms are thus simply the properties of exponents, since *a logarithm is always an exponent.*

I. *The logarithm of the base itself is unity,* since

$$b^1 = b \text{ or } \log_b b = 1.$$

II. *The logarithm of unity* (1) *is zero;* for

$$b^0 = 1 \text{ or } \log_b 1 = 0.$$

III. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let m and n be the factors and x and y their logarithms; then

$$b^x = m, \text{ or } \log_b m = x;$$
$$b^y = n, \text{ or } \log_b n = y.$$

We have

$$mn = b^x \cdot b^y = b^{x+y},$$

whence, by definition,

$$\log_b (mn) = \log_b m + \log_b n.$$

IV. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Using the same quantities as in III, we have

$$\frac{m}{n} = \frac{b^x}{b^y} = b^{x-y},$$

or

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n.$$

V. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power.*

Let

$$m = b^x, \text{ or } \log_b m = x;$$

then

$$m^n = (b^x)^n = b^{nx}, \text{ or } \log_b (m^n) = nx.$$

But

$$x = \log_b m;$$

hence

$$\log_b m^n = n \cdot \log_b m.$$

VI. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

Let

$$m = b^x, \text{ or } \log_b m = x;$$

then

$$\sqrt[n]{m} = m^{\frac{1}{n}} = b^{\frac{x}{n}},$$

or

$$\log_b (m^{\frac{1}{n}}) = \frac{\log_b m}{n}.$$

VII. *The logarithm of a proper fraction is always a negative number, if the base is a whole number.*

This follows at once from IV, where $\frac{m}{n}$ is a proper fraction if $m < n$; and we have $\log_b m - \log_b n$ negative.

VIII. *The logarithm of zero (0) is always negative infinity, if the base is a whole number.*

For we have

$$b^{-n} = \frac{1}{b^n}\Big]_{n=\infty} = 0$$

when n is indefinitely great, or, briefly, equal to infinity. That is,

$$\log_b 0 = -\infty.$$

Examples.

(To be worked without tables.)

1. Simplify $\frac{a^{\frac{1}{2}} \times a^{\frac{1}{3}}}{a^{\frac{1}{4}} \times a^{\frac{1}{5}}}$. *Ans.* $a^{\frac{7}{60}}$.

2. Check the result of Example 2 by evaluating each factor separately when $a=64$; when $a=81$.

3. Find the values of the following, the last two to three decimals:

$(343)^{-\frac{2}{3}}$, $(3125)^{\frac{2}{5}}$, $(81)^{-\frac{1}{4}}$, $(\frac{1}{10})^{-2}$, $(10)^{\frac{3}{2}}$, $(10)^{-\frac{1}{2}}$.

Ans. $\frac{1}{49}$, 125, $\frac{1}{3}$, 31.623, 0.316.

4. Find to three decimals $10^{.3} \times 10^{.6} \div 10^{.4}$. *Ans.* 3.162.

5. Given that $10^{.3} = 2.00$ to three significant figures, find $10^{.4}$ from the result of Example 4, and from $10^{.4} = 10 \div (10^{.3})^2$.

6. Find $\log_2 256$, $\log_3 243$, $\log_{10} .01$, $\log_a \frac{1}{a^4}$.

Ans. 8, 5, -2, -4.

7. Find $\log_{10^{-1}} 3$, $\log_{10^{-1}} \frac{1}{3}$, $\log_{10^{-1}}(-2)$.

Ans. 1000, $\sqrt[3]{10}$, 0.01.

8. Find the base if $\log_b 9 = \frac{2}{3}$, $\log_b 144 = 4$. *Ans.* 27, $2\sqrt{3}$.

9. Given $\log_{10} 2 = 0.30103$; find $\log_{10}$ of 62500, $\sqrt{1.25}$, .0025 and $\sqrt[3]{.0125}$. *Ans.* 4.79588, 0.048455, -2.60206, -0.63436.

10. Given $\log_{10} 2$ and $\log_{10} 3 = 0.47712$; find the $\log_{10}$ of 5.4, 0.5, 0.6. *Ans.* 0.73239, -0.30103, -0.22185.

11. With the data of Example 10, find $\log_{10}$ of 1080 and $(0.0025)^{\frac{1}{2}}$. *Ans.* 3.03342, -0.28912.

12. From same data find $\log_{10}$ of 12, 15, 18, 648 and 864.

Ans. 1.07918, 1.17609, 1.25527, 2.81157. 2.93651

20. The invention of logarithms is due to three men: Joost Bürgi of Switzerland, John Napier of Scotland, and Henry Briggs of England.

Between 1603 and 1611, Bürgi computed tables for facilitating computations, and in 1620 published them under the name of *Tables of Geometric Progressions.* These tables contained logarithms of numbers to the base 10. Napier, using the same fundamental principle as Bürgi, independently invented what were essentially logarithms of sines of angles, and published a table of them in 1614. Napier's logarithms had three defects: they were useful only in trigonometric computations, they decreased as the sines increased, and did not make $\log 1 = 0$. Briggs, with Napier's consent, remedied these defects and computed a table of logarithms to the base 10, such as we now use; it was completed by Adrian Vlacq, a Dutch bookseller, who published it for Briggs in 1628.

One other system of logarithms is in use, having the base approximately 2.71828; in many applications of mathematics, especially in the Calculus, such logarithms frequently appear; they are called *natural* or *hyperbolic* logarithms, in distinction from the common, or Briggs, logarithms. The natural logarithms were unknown to Napier, but are often called Napierian logarithms.

21. Common, or Briggs, Logarithms.—The advantage of this system is that the exact powers of 10 between which any number lies are easily found; thus:

$$\begin{aligned}
10^0 &= 1, & \log 1 &= 0;\\
10^1 &= 10, & \log 10 &= 1;\\
10^2 &= 100, & \log 100 &= 2;\\
10^3 &= 1000, & \log 1000 &= 3;\\
10^n &= 10^n, & \log 10^n &= n;\\
10^{n+1} &= 10^{n+1}, & \log 10^{n+1} &= n+1.
\end{aligned}$$

Also,

$$10^{-1}=.1, \quad \log 0.1 = -1;$$
$$10^{-2}=.01, \quad \log 0.01 = -2;$$
$$10^{-3}=.001, \quad \log 0.001 = -3.$$

It is evident from these numbers and their logarithms that the logarithm of any number which is not an exact power of 10 consists of a whole number followed by a decimal.

The integral part is called the *characteristic,* the decimal part the *mantissa.*

22. To Find the Characteristic of the Common Logarithm of a Whole Number.—It is also evident that *the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.*

For example,

$$\sqrt[3]{10}=2.1544$$

and

$$\log \sqrt[3]{10}=\tfrac{1}{3}\log 10=0.33333+;$$

thus,

$$\log 2.1544=0.33333,$$
$$\log 21.544=\log 2.1544+\log 10 =1.33333,$$
$$\log 215.44=\log 2.1544+\log 10^2=2.33333.$$

23. The Characteristic of the Logarithm of a Fraction.—The same process applied to decimal fractions, *numbers less than one,* gives

$$\log 0.21544 = \log 2.1544-\log 10 = 0.33333-1 = -0.66667,$$
$$\log 0.021544 = \log 2.1544-\log 10^2 = 0.33333-2 = -1.66667,$$
$$\log 0.0021544 = \log 2.1544-\log 10^3 = 0.33333-3 = -2.66667.$$

Negative logarithms are seldom written in that form, but the mantissa is made positive by the addition of 10 to the logarithm and placing -10 after it, the negative logarithms becoming

$$\log 0.21544 = -0.66667 = 9.33333 - 10,$$
$$\log 0.021544 = -1.66667 = 8.33333 - 10,$$
$$\log 0.0021544 = -2.66667 = 7.33333 - 10.$$

This device gives a *positive mantissa* to the logarithm of every *fractional number;* sometimes the notation $\bar{1}.33333$, for

$$-1+.33333,$$

is used, but it should always be avoided in computations.

The characteristic of the logarithm of a decimal fraction is found by subtracting from 9 the number of ciphers between the decimal point and the first significant figure, and writing -10 after the mantissa.

It is important to note that *the negative logarithm is the logarithm of the reciprocal of a number having the same logarithm taken positive.*

24. Many tables of common logarithms have been published, differing in the number of decimal places to which the mantissa is given, but substantially the same in arrangement. In all of them only the mantissa is given, as this is the same for all numbers having the same significant figures, the characteristic alone fixing the position of the decimal point. They range from four-place to ten-place tables, and assure results of computation only to as many significant figures as are given in the mantissas. The method of using them in practice will be described for Bowditch's Useful Tables (five figures).

25. To Find the Logarithm of a Number from the Tables.— The characteristic is determined by the rules already given; if the number consists of four figures or less, find the three left-

hand figures in the left-hand column of the tables; opposite this number and in the column headed by the fourth figure of the given number is the decimal part of the logarithm. Thus the logarithm of 3551 is 3.55035, and the logarithm of 3552 is 3.55047; the logarithm of any number between 3551 and 3552 will lie between these two logarithms, and within such small limits the increase in the logarithm is proportional to the increase in the number. Thus the logarithm of 3551.5 is midway between the logarithms of 3551 and 3552, or it is found by adding $\frac{5}{10}$ of the difference between the logarithms to the lesser one; thus log $3551.5 = 3.55035 + \frac{5}{10}$ of $.00012 = 3.55041$; similarly log $3551.3 = 3.55035 + \frac{3}{10}$ of $.00012 = 3.55039$; log $355.125 = 2.55035 + \frac{25}{100}$ of $.00012 = 2.55038$. The operation of finding the logarithm of a number between those given in the table is called *interpolation;* it is facilitated by small tables placed at the side of the page, called *tables of proportional parts.* On the page of the tables we have been using there is a small table headed 12, which is the difference between the logarithms of 3551 and 3552. In this table, opposite the fifth figure of the given number, we find the amount to be added to the logarithm of the first four figures. The logarithms are never to be carried beyond five figures. If in the interpolation we obtain a sixth figure, we neglect it if it is less than 5, and add one to the fifth figure if the sixth figure is 5 or greater. If the given number has less than three digits its logarithm may be found from the first page of the tables, but it is often more convenient to look for the logarithm of the same number with ciphers annexed.

26. To Find the Number Corresponding to a Given Logarithm (its Antilogarithm).—With the *mantissa* of the given logarithm as a guide, take from the tables the number corresponding to the *nearest smaller* mantissa; note the difference between the given and the tabular mantissa, and also the difference between the latter and the next higher tabular mantissa.

The ratio of the first difference to the tabular difference is the *proportional part* of 10 to annex to the four figures of the tabular antilogarithm already taken out. Then point off the number according to its characteristic.

If a whole number, its integral places are one greater than the characteristic of the given logarithm.

If a decimal, the number of ciphers after the decimal point is indicated by the difference between the given characteristic and 9.

For example, required the antilogarithm of $9.55037-10$.

The next lower tabular mantissa is .55035, corresponding to 3551; the difference between this mantissa and the given one is .00002, while the tabular difference is .00012; hence the interpolation is $\frac{2}{12}$ of $10=1.6$ or 2, making the figures of the antilogarithm 35512. From the given characteristic the required number is 0.35512.

27. Tables for Interpolation.—The work of interpolation is greatly facilitated by marginal tables giving the proportional parts of each tabular difference corresponding to the *fifth* place of the number, when taking out the logarithm of a number. They are, of course, used in the reverse way when seeking the fifth place of the number from the mantissa. The tabular differences near the beginning of the table are large, and the logarithm of a number of six figures may be obtained; but when the differences are small, there is usually no gain in the accuracy of the five-figure logarithm by taking account of the sixth significant figure.

28. The Arithmetical Complement.—The logarithm of the reciprocal of a number is called the *arithmetical complement* of the logarithm of the number, and is usually written *colog*. This device is used to convert a divisor into a multiplier. Thus, if

we have to find the logarithm of $\frac{abc}{d}$, we know that $\log\left(\frac{1}{d}\right)$ $=\log 1-\log d=0-\log d=\operatorname{colog} d$; therefore we write

$$\log\left(\frac{abc}{d}\right)=\log a+\log b+\log c+\operatorname{colog} d.$$

If $\log d=3.57934$, $\quad\operatorname{colog} d=6.42066-10.$

If $\log d=9.31820-10$, $\quad\operatorname{colog} d=0.68180.$

The colog is most conveniently obtained by subtracting each figure of the logarithm from 9, except the last significant figure, which is subtracted from 10. The subtraction should begin with the left-hand figure, thus writing the colog from left to right.

Exercises.

Verify the following:

log	0.0075329	=	7.87696−10.
log	3426	=	3.53479.
log	7.854	=	0.89509.
log	3.1416	=	0.49715.
log	2.71828	=	0.43429.
log	30.473	=	1.48391.
log	10.817	=	1.03411.
log	9.9999	=	1.00000.
log	102.01	=	2.00864.
log	0.36413	=	9.56126−10.
$\log^{-1}$	0.85670	=	7.1895.
$\log^{-1}$	3.67799	=	4764.2.
$\log^{-1}$	7.51322−10	=	0.00326.
$\log^{-1}$	9.77777−10	=	0.59947.
$\log^{-1}$	0.00215	=	1.005.
$\log^{-1}$	2.18562	=	153.33.
$\log^{-1}$	6.98354−10	=	0.0009628.
$\log^{-1}$	0.99992	=	9.9982.
$\log^{-1}$	8.66767−10	=	0.046523.
$\log^{-1}$	1.77236	=	59.205.

29. Examples Illustrating the Use of Logarithms.—In practical work the subscript 10, denoting the base, is omitted. It is of the greatest importance that the work of computing by logarithms be arranged in a convenient form. The following examples are models which are to be strictly followed:

Multiplication.—Evaluate $272 \times 0.31734 \times 0.02165 \times 6.4335$.

272	log 2.43457	(Read: 272; its log is 2.43457.)
0.31734	log 9.50153 − 10	Note that the sum of the characteristics is 21 − 20, or 1.
0.02165	log 8.33546 − 10	
6.4335	log 0.80845	
12.023	log 1.08001	

Ans.

Division.—Evaluate $8.6302 \div 9.7124$.

8.6302	log 0.93602
9.7124	log 0.98733
0.88856	log 9.94869 − 10

Roots.—Evaluate $\sqrt[3]{0.17638 \times 2.1279}$.

	0.17638	log 9.24645
	2.1279	log 0.32795
x^3		log 29.57440 − 30
x	0.72132	log 9.85813 − 10

Here the characteristic of x^3 is written 29 − 30 so that the logarithm of x shall be in the proper form for immediate reference to the tables.

Powers.—Evaluate $\left[\frac{19.876}{38.345}\right]^2$.

	19.876	log 11.29833 − 10
	38.345	log 1.58371
$\sqrt{x}$		log 9.71462 − 10
x	.26868	log 9.42924 − 10

Here the characteristic 19 − 20 is evidently the same as 9 − 10.

It is sometimes valuable to note that a sixth figure of a number very near the beginning of the table affects in some cases the last figure of the mantissa; e. g., 110.215 and 110.224, though each is written 110.22 as a five-figure number, have logarithms 2.04224 and 2.04228 respectively. It is even sometimes worth while to determine a sixth figure from a mantissa in this part of the table, for although it is not itself closely accurate, it may improve the accuracy of subsequent operations.

Examples.

Find the value of each of the following expressions:

1. 3.1416×2.7183, $29.572 \times .00036841$, and $335000000 \times .000099854$. *Ans.* 8.5398, 0.0108945, 33451.

2. $.00035842 \times 27925$, $.0066666 \times .33333$, and $52564 \times .0082546$. *Ans.* 10.009, 0.0022222, 433.90.

3. $2727.5 \times .37375$, 511.01×100.09, and 263.58×21.542. *Ans.* 1019.4, 51147, 5678.1.

4. $\frac{2.9275 \times 34.278}{505.92}$, and $\frac{.0031593 \times 684.82}{.00096548}$. *Ans.* 0.19835, 2240.9.

5. $\frac{.00037562 \times 645.29}{125.82}$, and $\frac{2625.4 \times .000055555}{31956}$. *Ans.* 0.0019264, $\frac{4.5641}{10^6}$.

6. $\frac{3.8434 \times .0029825}{.037378}$, and $\frac{48.962 \times 39.595}{78.545}$. *Ans.* 0.30667, 24.682.

7. $\frac{9999.5 \times .0077764}{2842.1}$, and $\frac{.02542 \times .000063845}{911.25}$. *Ans.* 0.027360, $\frac{1.7810}{10^9}$.

8. $\frac{2964.5 \times 38.423}{75.65 \times 84.384}$, and $\frac{2954.5 \times 64.532}{911.36 \times 318.5}$. *Ans.* 17.843, 0.65684.

9. $\frac{411.44 \times 38.456}{2928.5 \times .07562}$, and $\frac{.006815 \times 992.85}{445.28 \times .000056842}$. *Ans.* 71.447, $267.3\overline{4}$.

10. $\frac{26.893 \times .0000545}{319.62 \times .00068432}$, and $\frac{75859 \times .0028242}{37568 \times .09185}$.

Ans. 0.0067010, 0.062086.

11. $(12)^9$, $(1.5)^{15}$, $(1.045)^{26}$, $(1.05)^{49}$.

Ans. 5.1596×10^9, 437.88, 3.1414, 10.922.

12. $(14\frac{3}{4})^8$, $(7\frac{2}{3})^{10}$, $\sqrt{10}$, $\sqrt{36443}$.

Ans. 2.2404×10^9, 7.0162×10^8, 3.1623, 190.90.

13. $\sqrt{54.9}$, $\sqrt[3]{31}$, $\sqrt[4]{90}$, $\sqrt[5]{10.974}$.

Ans. 7.4095, 3.1414, 4.4814, 2.2223.

14. $(7)^{\frac{1}{2}}$, $(442)^{\frac{1}{3}}$, $(1.03)^{\frac{1}{4}}$, $(50.8)^{\frac{1}{5}}$.

Ans. 1.3205, 1.9676, 1.0006, 12.177.

15. $(.00625)^{\frac{1}{4}}$, $(.37568)^{\frac{1}{2}}$, $(442.84)^{\frac{1}{3}}$, $(.03628)^{\frac{1}{5}}$.

Ans. 0.63041, 0.89694, 5.2686, 0.51514.

16. $(469.82)^{-\frac{1}{2}}$, $(665.35)^{-\frac{1}{3}}$, $(33.982)^{-\frac{1}{4}}$, $(.058086)^{-\frac{1}{5}}$.

Ans. 0.50480, 0.39510, 0.24406, 1.7668.

17. $(.00062584)^{\frac{1}{2}}$, $(.35694)^{\frac{1}{3}}$, $(.0022893)^{\frac{1}{4}}$, $(.055642)^{\frac{1}{5}}$.

Ans. 0.39770, 0.59744, 0.13180, 0.61787.

18. Find the eighth root of $\frac{a^5b^3c^2d^{\frac{1}{2}}}{e^2f^3g^4}$; given a, 301.03; b, .00036954; c, .0028182; d, 35890000; e, .000002814; f, 561.29; g, 2718.3. *Ans.* 0.02838.

19. Find the seventh root of $\frac{abcd}{efg}$; given a, .003589; b, .00015785; c, 3401.5; d, 15.785; e, 22.322; f, 25.79; g, 25790000. *Ans.* 0.02139.

30. Operations with Negative Numbers.—Although negative numbers have no logarithms, we may treat them as if they were positive in processes involving multiplication, division, involution or evolution, and determine the *sign* of the result by the ordinary rules of algebra. The logarithms of such numbers occurring in practice are designated by the letter n after the logarithm.

Example: Find the value of $\frac{a^3b^2c}{d^{\frac{1}{2}}}$, given

a, $-.00275$; b, -100.5; c, -5075.5; d, $-.001875$.

The work is arranged as follows:

$a,-$	.00275	$\log 7.43933-10n$	$3\log 2.31799-10n$
$b,-$	100.5	$\log 2.00217n$	$2\log 4.00434$
$c,-$	507.5	$\log 2.70544n$	$\log 2.70544n$
$d,-$	.001875	$\log 7.27300-10n$	$\frac{1}{3}\operatorname{colog} 0.90900n$
$x,-$	.86452		$\log 9.93677-10n$

Here we give the negative sign to the result because the last column contains an *odd* number of negative factors.

31. To Find the Logarithm of a Number to a New Base when the Logarithms to another Base are Known.—Given $\log_b n$, required $\log_a n$. Let $x=\log_a n$, or $a^x=n$.

Take logarithms of both sides of the equation to the given base b.

$$x\log_b a=\log_b n,$$

$$x=\frac{\log_b n}{\log_b a}=\log_a n.$$

If in this equation we let $n=b$, it becomes

$$\frac{1}{\log_b a}=\log_a b, \text{ or } \log_a b\times\log_b a=1.$$

32. *Whenever the logarithm of a fraction is to be multiplied or divided by a decimal, we must first express the logarithm in its negative form.*

This becomes apparent in such examples as the following *exponential* equation:

Find x from the equation $(.37544)^x=5.8592$. Taking the logarithms of both members of the equation,

$$x\log(.37544)=\log(5.8592).$$

$$x=\frac{\log(5.8592)}{\log(.37545)}=\frac{0.76784}{9.57455-10}=\frac{0.76784}{-(.42545)}.$$

Reducing this fraction by ordinary division, we find

$$x = -1.8048;$$

or we may perform the division by logarithms as follows:

$$\begin{array}{rl} .76784 & \log 9.88527 - 10 \\ - \ .42545 & \log 9.62885 - 10n \\ \hline x = -1.8048 & \log 0.25642n \end{array}$$

The quantity $\log .37545 = 9.57455 - 10$ is the indicated difference between two numbers, and the subtraction must be performed before the fraction can be reduced.

Examples.

1. Find x from $20^x = 100$. *Ans.* $x = 1.5372$.
2. Find x from $5^{2x} = 317.46$ *Ans.* $x = 1.7895$.
3. Solve the equation $7^x = 100$. *Ans.* 2.3666.
4. Solve the equation $(1.05)^x = (8.25)^{\frac{1}{x}}$. *Ans.* 6.5764.
5. Solve the equation $5^{\frac{1}{x}} = 1.307$. *Ans.* 6.0110.
6. Solve the equation $3^{x-4} \times 5^{3x} = 7^{2x-5}$.
Ans. $x = -.26216$.
7. Find the base of a system of logarithms in which $\log 2 = 0.69315$. *Ans.* 2.7183.
8. Find x from $a = e^{cx}$; given a, 375850; c, 1.5708; e, 2.7183.
Ans. 9.6383.
9. Find the fifth root of $\left\{\frac{a^2bc^3}{de} + f\right\}$; given a, .032713; b, 643.68; c, .99985; d, 68432; e, .000835; f, .0005.
Ans. 0.41660.
10. Find $\log_{12} 159$, and $\log_{159} 12$. *Ans.* 2.0399, 0.49022.
11. Find the seventh root of $\frac{a^3bc^2d^{\frac{1}{2}}}{e^5g}$; given a, .087295; b, 310.85; c, 41192; d, .000086584; e, 39.644; g, 100.84.
Ans. 0.31825.
12. Find K; given $s = \frac{1}{2}(a+b+c+d)$,
$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$,
a, 6.3246; b, 7.7459; c, 8.5441; d, 5.1961. *Ans.* 46.693.

13. Find the value of x^{π} when $x=0.37544$. *Ans.* 0.69225.

14. Find the value of e^{π}, and π^{e}, when $e=2.71828$, $\pi=3.1416$.
Ans. 23.139, 22.458.

15. Find the value of $e^{\frac{1}{\pi}}$, $\pi^{\frac{1}{e}}$, $e^{-\frac{1}{\pi}}$, and $(\pi)^{-\frac{1}{e}}$.
Ans. 1.3748, 1.5237, 0.20788, 0.65631.

16. Find the value of x^{e} when $x=2.5062$. *Ans.* 10.000.

17. Find the value of x^{10x} when $x=0.72923$.
Ans. 0.10000.

18. Find the value of $x^{\frac{1}{e}}$ when $x=0.399015$.
Ans. 0.10000.

19. Find the value of $x^{\frac{10}{\pi}}$ when $x=1.3713$.
Ans. 10.000.

20. If a regular tetraedron is inscribed in a sphere of radius a, a sphere inscribed in the tetraedron, a regular tetraedron in the second sphere, and so on, the volume of the nth tetraedron is $\frac{8a^3\sqrt{3}}{(27)^n}$. Show that if $a=4000$ mi., the volume of the 18th tetraedron is 3.8791 cu. in., and compute the volumes in cubic inches of the 15th, 19th, 20th and 25th tetraedra.
Ans. 76352, 0.14367, 0.0053211, $\frac{3.7084}{10^{10}}$.

21. Find the length in feet of one minute of arc on a sphere of 3958.4 mi. radius. *Ans.* 6079.7.

22. Given $a=3.7124$, $b=32.617$, find $a+b$, $a-b$, $\frac{a}{b}$, ab, $\log a+\log b$, $\log(a+b)$, $\log a-\log b$, $\log(a-b)$, $\log a\times\log b$, and $\frac{\log a}{\log b}$.

23. Given $a=0.37124$, $b=0.32617$, find the same functions of a and b as in Example 22.

24. What sum will amount to \$1000 in ten years, interest being at 3½ per cent per annum, compounded semi-annually?
Ans. \$706.82.

25. In how many years will a sum double itself at interest as in Example 24? *Ans.* Twenty years.

26. Given that if the sides of a plane triangle are a, b, c, and $s=\frac{1}{2}(a+b+c)$, the area is $A=\sqrt{s(s-a)(s-b)(s-c)}$, find the

area and the shortest of the three altitudes of a plane triangle of which the sides are 147.73, 159.09, 170.45.

Ans. $A=10847$, $h_c=127.275$.

27. Noting that an expression of the form $\sqrt{m^2-n^2}$ should always be treated as $\sqrt{(m-n)(m+n)}$ for purposes of computation, find from the result of Example 26 the lengths of the segments into which the computed altitude divides the longest side. *Ans.* 75.002, 95.449.

33. Trigonometric Functions.—Among the simple computations of frequent occurrence are the evaluations of the forms $\sqrt{a^2-b^2}$, $\sqrt{a^2+b^2}$, $\sqrt{a-b}$ and $\sqrt{a+b}$, where a and b are given numbers or products of given factors.

Such forms are often greatly simplified by the use of *trigonometric functions or ratios,* which are defined as follows:

If from any point in one side of an angle a perpendicular be drawn to the other side, then, in the right triangle thus formed:

The *sine* of the angle is the ratio of the *perpendicular* to the *hypothenuse.*

The *cosine* of the angle is the ratio of the *base* to the *hypothenuse.*

The *tangent* of the angle is the ratio of the *perpendicular* to the *base.*

The *cosecant, secant* and *cotangent* are the *reciprocals,* respectively, of the *sine, cosine* and *tangent.*

These six ratios are the same for the same angle, no matter of what size the triangle is drawn; for all right triangles with the same acute angle are similar, and the ratios of their homologous sides are equal.

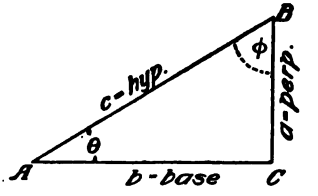


FIG 1.

The functions are shown graphically, with their abbreviations,

in the following scheme, in which the sides of the triangle are represented by a and b, and the hypothenuse by c.

$$\sin\theta=\frac{a}{c},\ \cos\theta=\frac{b}{c},\ \tan\theta=\frac{a}{b},$$

$$\operatorname{cosec}\theta=\frac{c}{a},\ \sec\theta=\frac{c}{b},\ \cot\theta=\frac{b}{a}.$$

If the angle $\theta>90°$, the perpendicular falls on the other side produced, and the base, b, is negative; thus all the trigonometric ratios, except the sine and consecant, are *negative* for angles in the second quadrant.

From the relation of the sides in a right-angled triangle $a^2+b^2=c^2$, we deduce the fundamental equations:

$$\frac{a^2}{c^2}+\frac{b^2}{c^2}=1, \text{ or } \sin^2\theta+\cos^2\theta=1;$$

$$\frac{a^2}{b^2}+1=\frac{c^2}{b^2}, \text{ or } \quad 1+\tan^2\theta=\sec^2\theta.$$

A book of logarithmic tables usually includes a table of the numerical values of these trigonometric ratios and also a table of their logarithms. The latter is used just as a table of logarithms of numbers, and the interpolation is made on the same principle. In Bowditch's tables the logarithms are given for intervals of one minute of arc, and the "proportional part" is given for the seconds in annexed tables. Thus, if we wished to find the log sine of 30° 27′ 32″, we would take from the tables for 30° 27′ the logarithm 9.70482−10, and add to its mantissa $\frac{32}{60}\times 22$ (the tabular difference for 1′), or 12. The log sine of the given angle would thus be 9.70494−10.

The evaluation of $\sqrt{a^2+b^2}$ would be carried out as follows:

$$\sqrt{a^2+b^2}=b\sqrt{1+\frac{a^2}{b^2}};$$

let

$$\tan\theta=\frac{a}{b},$$

then

$$\sqrt{1+\frac{a^2}{b^2}}=\sqrt{1+\tan^2\theta}=\sec\theta.$$

Finally

$$\sqrt{a^2+b^2}=b\sec\theta.$$

The numerical work, where $b=76.371$, $a=70.832$, is given:

a,	70.832	log 1.85023	
b,	76.371	log 1.88293	log 1.88293
θ,	42° 50′ 42″	log tan $\overline{9.96730}-10$	log sec 0.13478
$\sqrt{a^2+b^2}=104.18.$			log $\overline{2.01771}$

Exercises.

Verify the following:

log sin 19° 18′ 25″ = 9.51934 − 10.
log cos 19° 20′ 45″ = 9.97476 − 10.
log tan 19° 57′ 18″ = 9.56001 − 10.
log sec 64° 42′ 20″ = 0.36930.
log csc 48° 48′ 48″ = 0.12345.
log cot 35° 17′ 13″ = 0.15015.
log tan 58° 28′ 02″ = 0.21212.
sin 19° 18′ 25″ = 0.33063.
cos 64° 42′ 20″ = 0.42727.

9.63721 − 10 = log sin 25° 42′ 14″.
9.56278 − 10 = log cos 68° 34′ 02″.
0.37521 = log tan 67° 08′ 42″.
9.66666 − 10 = log cot 65° 06′ 05″.
0.12439 = log sec 41° 19′ 38″.
0.02524 = log csc 70° 39′ 15″.
9.63070 − 10 = log sin 25° 17′ 40″.
0.42727 = sin 25° 17′ 40″.
0.75095 = cos 41° 19′ 38″.

Show by computation that $\frac{\pi}{2}=\sec 45^\circ\times\sec\frac{45^\circ}{2}\times\sec\frac{45^\circ}{4}\times\sec\frac{45^\circ}{8}\times\sec\frac{45^\circ}{16}\times\sec\frac{45^\circ}{32}\times\sec\frac{45^\circ}{64}\times\sec\frac{45^\circ}{128}$; given, $\pi=3.1416$.

Examples.

1. Given $a=76.371$, $b=70.382$, $c=13.204$, evaluate $\sqrt{b^2+4ac}$.
Ans. 94.800.

2. With the data of Example 1, evaluate $\sqrt{b^2-4ac}$, using $\sqrt{1-\sin^2\theta}=\cos\theta$. *Ans.* 30.332.

3. With same data, evaluate $\sqrt{a^2-b^2}$ by the formula
$$\sqrt{a^2-b^2}=\sqrt{(a-b)(a+b)}.$$ *Ans.* 29.646.

4. Evaluate $\sqrt{a^2-b^2}$ by trigonometric formulæ.

CHAPTER III.

Functions and Equations.

34. Functions.—Any expression which involves a quantity x, and whose value depends upon the value of x, is called a *function of* x.

Thus, $\sqrt{x^2-9}$, a^x, $\log(1+x)$, $\sin x$ are functions of x; the length of a chord of a circle of given radius a is a function of its distance, x, from the center; this function is $2\sqrt{a^2-x^2}$.

In studying a function, we give such values as we choose to the quantity on which it depends, and note the resulting change in the function; the function is consequently called the *dependent variable*, and the quantity on which it depends is called the *independent variable*. The independent variable is also called the *argument*.

A quantity which does not vary is called a *constant;* for instance, if through a point inside of a circle a chord be drawn and then revolved about the point, the lengths of its segments may vary, but the product of these lengths is constant.

The same quantity may be constant in one problem and variable in another. Thus, when we consider how the function ax^2+bx+c changes, a, b and c being given values and x varying, a, b, and c are constants; but when we study the change in one of the roots of $ax^2+bx+c=0$ due to changes in a, b, and c, a, b, and c are variables.

The symbol $f(x)$ is used to abbreviate the expression "function of x" or to stand for a designated function of x. When more than one function of x is to be designated, other similar expressions are used, such as $f_1(x)$, $f'(x)$, $F(x)$, $Q(x)$, $\phi(x)$, etc.

The result of substituting a given quantity for the independent variable in a function is represented by replacing the variable by the given quantity in the symbol for the function; thus, if we designate $\sqrt{3x^2+a^2}$ by $f(x)$, $f(y)$ designates $\sqrt{3y^2+a^2}$, and similarly

$$f(b)=\sqrt{3b^2+a^2},$$
$$f(a)=\sqrt{3a^2+a^2}=2a,$$
$$f(0)=\sqrt{0+a^2}=a.$$

When it is better to express the function without abbreviation, these same relations are written in the form:

$$\sqrt{3x^2+a^2}]_{x=y}=\sqrt{3y^2+a^2},$$

read as follows: $\sqrt{3x^2+a^2}$, when $x=y$, is equal to $\sqrt{3y^2+a^2}$.

The connection between the two methods of notation may be expressed:

$$f(a)=[f(x)]_{x=a}.$$

The same functional notation may be extended to expressions involving two independent variables; thus,

$$f(x,y)=\sqrt{x^2+y^2-a^2}$$

is a function of x and y.

An *algebraic function* is one in which the variables are connected only by algebraic operations and have only constants for exponents.

If the exponents of the variables are integers, the function is called a *rational* function; and if the variables do not appear as denominators of fractions, the function is called *integral.*

A rational integral algebraic function with rational integral coefficients is a *polynomial.*

The *degree* of a polynomial is the degree of the highest power of the independent variable occurring in the polynomial.

A polynomial of the first degree is called *linear;* a polynomial of the second degree, *quadratic;* a polynomial of the third degree, *cubic.* $2x+3$, $4x^2+5x-1$, $7x^3-3x^2+2x+4$ are examples of linear, quadratic, and cubic polynomials, respectively.

Examples.

1. Given $f(x)=x^3-2x^2+3x+1$, find or write the values of $f(y), f(a), f(1), f(0), f(a+b)$.
2. Given $\phi(x)=a^x$, find or write $\phi(1)$, $\phi(0)$, $\phi(-2)$, $\phi(x+y)$.
3. Given $\phi(x)=a^x$, show that $\phi(x+y)=\phi(x)\times\phi(y)$.

35. Equations and Identities.—An assertion of equality between two functions is called an *equation.*

An equation may always be reduced to a form in which one member is zero, and thus if it involve only one variable, x, may be represented in general by $f(x)=0$.

If $f(x)$ is a rational integral algebraic function of x, $f(x)=0$ is an *algebraic equation.*

An equation which is true for only a limited number of values of the variable involved is called a *conditional equation.* Thus,

$$6x^2+x-2=0$$

is true only when $x=\frac{1}{2}$ or $-\frac{2}{3}$.

An assertion of the equivalence of two forms of expression of the same function is called an *identity,* or an *identical equation.*

An *identical equation,* commonly called an *identity,* is true for every possible value of the variable involved. Thus,

$$(9-x^2)\equiv(3-x)(3+x)$$

is true for any value of x whatever.

The sign $\equiv$ is sometimes used in an identical equation and is read "is identically equal to" or "is identical with."

An *equation* presents a problem; and if it is put, as it always may be, in the form $f(x)=0$, the problem of the equation consists in finding what values of x make $f(x)$ have the value zero.

An *identity* asserts a general truth. For instance, the length of a chord at the distance x from the center of a circle of radius a is $2\sqrt{a^2-x^2}$. If we wish to solve the problem of finding the distance from the center if the chord is equal in length to the radius, we write the conditional equation $2\sqrt{a^2-x^2}=a$, which we find is satisfied by the value $x=\frac{a}{2}\sqrt{3}$. On the other hand, the identical relation $2\sqrt{a^2-x^2}\equiv 2\sqrt{(a-x)(a+x)}$ may be used in computing the length of the chord drawn at a given distance x from the center, no matter what x may be.

36. Division of Equations and Identities.—It should be noted that the two members of an identity are two expressions for the same thing, differing merely in appearance, so that if the same operation is performed on both members, the two results are identical. For instance, if we divide the identity

$$(x^2-9)(x+3)\equiv(x+3)^2(x-3)$$

by $(x-3)$, the result,

$$(x+3)(x+3)\equiv(x+3)^2,$$

is still an identity.

If any factor involving the unknown quantity is divided out of a *conditional equation* however, some or all of the possibility of solving it will have been lost. For instance, if we divide the equation

$$(x-3)(x+7)=(x-3)(x+9)$$

by $(x-3)$, the resulting equation,

$$x+7=x+9,$$

will not be true for any value of x, although the original equation holds when $x=3$; and again, if we divide

$$(x-3)(2x+7)=(x-3)(x+9)$$

by $(x-3)$, the resulting equation is true only when $x=2$, although the original equation holds for $x=2$ or for $x=3$.

It is to be remembered, then, that:

I. If a factor is divided out of an identity, the result is an identity; but

II. If a factor is divided out of a conditional equation, the result may or may not be true; and for a complete solution of the original equation, we must solve not only the quotient-equation, but also the equation formed by equating the factor to zero.

An identity of which each member is identically zero cannot be divided by an expression which is identically zero; this is the only case of an operation not leading to a single definite result that we ordinarily have to guard against.

Examples.

Determine whether each of the following relations is an identity or an equation, and if an equation, for what values of x it is true:

1. $(x-3)^3=x^3-27$.
2. $(x-3)^3=9x(3-x)+x^3-27$.
3. $(a^2-ax+5)(a^2+2ax-7)=a^4+3a^3-20a^2+51a-35$, a being constant.

Ans. An equation which is true when $x=3$ or $x=\dfrac{a^2-6a+17}{2a}$.

4. $(x^2-3x+5)(x^2+6x-7)=x^4+3x^3-20x^2+51x-35$.
5. $\sqrt{\dfrac{a-b}{a+b}}=\dfrac{\sqrt{a^2-b^2}}{a+b}$, a and b variable.
6. $(x-7)(x^2-6x+8)=(x^2-9x+14)(x-4)$.
7. $(x-5)(x^2+2x-8)=(x^2-3x-10)(x-4)$.

IMPORTANT FUNCTIONS AND IDENTITIES.

37. Arithmetic Progressions.—A succession of numbers, each of which is formed by *adding* a given number to the one next before it, is called an *arithmetic progression*. The given number thus added is called the *common difference* of the progression; it may be either positive or negative.

For example:

$$3, 5, 7, 9, 11, 13, \text{ etc.},$$

form an arithmetic progression of which the *first term* is 3 and the common difference is 2;

$$5, \tfrac{9}{2}, -2, -\tfrac{11}{2}, -9, -\tfrac{25}{2}, \text{ etc.},$$

form an arithmetic progression of which the first term is 5 and the common difference is $-\frac{7}{2}$.

In case a limited number of terms is considered, the *number of terms* and the *last term* are also spoken of, and the terms between the first and last are called the *arithmetic means* between them. When there are only three terms, the middle one is called the *arithmetic mean* or *average* of the first and last, and is equal to one-half their sum.

The symbols $T_1, T_2, T_3, \ldots, T_n$ are used to indicate the first, second, third, ..., nth terms of the progression, and the symbol S_n to indicate the sum of the first n terms. If the progression has just n terms, T_n is the last term, and S_n the sum of all the terms.

38. General Term and Last Term.—In any arithmetic progression, if the first term is a, and the common difference d, we have by definition:

$$T_1=a,\ T_2=a+d,\ T_3=a+2d,\ T_4=a+3d, \text{ etc.}$$

It is evident that for any value of n, the nth term is

$$T_n=a+(n-1)d. \qquad (1)$$

This is called the *expression for the general term* of the progression. When the number of terms is limited, this number is usually denoted by n, and the last term by l; so that the formula for the last term is

$$l=a+(n-1)d. \qquad (2)$$

For example, in the first of the series above, $T_n=1+2n$; for $n=1, 3, 7, 10$, this gives $T_1=3$, $T_3=7$, $T_7=15$, $T_{10}=21$; if

we suppose the series to contain n terms, it also gives the value of the last term, $l=1+2n$.

39. Sum of n Terms.—Using a, d and l as above, we have:

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + (a+3d) + \ldots + (l-d) + l \\ S_n &= l + (l-d) + (l-2d) + (l-3d) + \ldots + (a+d) + a \\ \hline 2S_n &= (a+l) + (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) \end{aligned}$$

The terms as so paired, by writing S_n both backward and forward, that the sum of the two in each of the n pairs is $(a+l)$; hence

$$S_n = \frac{n}{2}(a+l). \tag{3}$$

Equations (2) and (3) involve the five quantities a, d, n, l, S_n, and furnish the means, if three of these quantities are given, of finding the other two. In particular, we get by eliminating l from (2) and (3):

$$S_n = \frac{n}{2}[2a+(n-1)d]. \tag{4}$$

Formula (4) may be used as an expression for the sum of all the terms of a series, or as an identity giving the sum of any number of terms; for the second purpose it is more conveniently written

$$S_n \equiv \tfrac{1}{2}[(2a-d)n+dn^2]. \tag{5}$$

In the series 3, 5, 7, 9, 11, etc., the sum of any n terms beginning with the first is

$$S_n \equiv 2n+n^2.$$

Such an identity will serve to give the series completely, for it gives

$$S_1 = T_1, \quad S_2 = T_1 + T_2,$$

from which the first two terms can be found, and hence the whole series.

Examples.

1. Find from (4) the value of n in terms of a, d, and S_n.

Ans. $\frac{1}{2d}[d-2a\pm\sqrt{8dS_n+(2a-d)^2}]$.

2. How many terms of 12, 16, 20, ... must be taken to make the sum 208? *Ans.* 8.

3. How many terms of 3, $\frac{9}{2}$, 6, ... must be taken to make the sum $448\frac{1}{2}$? *Ans.* 23.

4. Find the expressions for T_n and S_n of the series of Examples 2 and 3. *Ans.* $T_n=8+4n$, $S_n=2n(n+5)$ and

$$T_n=\tfrac{3}{2}(n+1),\ S_n=\frac{3n}{4}(n+3).$$

5. Given $T_n=3n-1$, find the series and the expression for S_n. *Ans.* $S_n=\frac{n}{2}(3n+1)$.

6. Given $S_n=3n^2+2n$, find the series and the expression for T_n. *Ans.* $T_n=6n-1$.

7. Insert ten arithmetic means between 243 and 23.

Ans. 223, 203, ..., 43.

8. The sum of three numbers in arithmetic progression is 18 and their product is 192; what are the numbers? *Ans.* 4, 6, 8.

9. Show that the sum of the first n integers is $\frac{n}{2}(n+1)$.

10. In an arithmetic progression $T_4=3$, $T_{16}=-13$. Find T_n, S_n, and the series. *Ans.* Series, 7, $\frac{17}{8}$, $\frac{13}{8}$, ...

11. In an arithmetic progression $T_6=2$, $S_5=-11$. Find T_n, S_n, and the series. *Ans.* Series, -5, $-\frac{18}{5}$, $-\frac{11}{5}$, ...

40. Geometric Progressions.—A succession of numbers, each of which is formed by *multiplying* the one next before it by a given number, is called a *geometric progression.* The given multiplier is called the *common ratio* of the progression.

For example:

3, 6, 12, 24, etc.,

form a geometric progression of which the *first term* is 3 and the common ratio is 2;

$$6,\ -4,\ \tfrac{8}{3},\ -\tfrac{16}{9},\ \tfrac{32}{27},\ -\tfrac{64}{81},\ \text{etc.},$$

form a geometric progression of which the first term is 6 and the common ratio is $-\frac{2}{3}$.

The expressions *number of terms, last term,* and *means* are used in connection with geometric progressions as with arithmetic. The geometric mean between two numbers is also called their *mean proportional,* and is equal to the square root of their product.

41. General Term and Last Term.—In any geometric progression, if the first term is a, and the common ratio r, we have by definition:

$$T_1=a,\ T_2=ar,\ T_3=ar^2,\ T_4=ar^3,\ \ldots,\ T_n=ar^{n-1}.$$

In a geometric progression of a limited number, n, of terms, the last term, l, is written:

$$l=ar^{n-1}. \tag{1}$$

Multiplying S_n by r and subtracting the product from S_n, we have

$$\begin{aligned} S_n &= a+ar+ar^2+ar^3+\ldots+ar^{n-1} \\ r\cdot S_n &= \quad ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n \\ \hline (1-r)S_n &= a-ar^n \end{aligned}$$

whence

$$S_n=a\,\frac{1-r^n}{1-r}. \tag{2}$$

Since $ar^{n-1}=l$, this is also written:

$$S_n=\frac{a-rl}{1-r}. \tag{3}$$

In case r is greater than 1, it is more convenient to write

$$S_n=a\frac{r^n-1}{r-1}=\frac{rl-a}{r-1}.$$

Equations (1) and (2) furnish the means, when any three of the quantities a, r, n, l, S_n are given, of finding the other two.

Examples.

1. Find the sum of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{16}$, etc., to five terms. *Ans.* $\frac{781}{768}$.

2. The sum of three numbers in geometric progression is 26, their product 216; what are the numbers? *Ans.* 2, 6, 18.

3. Find S_n for the progression 6, -3, $+\frac{3}{2}$, etc.

$$Ans. \quad \frac{2^n+(-1)^{n+1}}{2^{n-2}}.$$

4. Given that in the series 5, 20, 80, ..., $S_n=109225$, find n.

$$Ans. \quad n=\frac{\log 65536}{\log 4}=8.$$

42. The Infinite Geometric Progression.—In a geometric progression, if the common ratio is greater than unity ($r>1$) or less than negative unity ($r<-1$), the successive terms are increasingly greater in numerical value; if $r=1$, they are all equal; if $r=-1$, they are numerically equal, alternating in sign; and if the common ratio is a positive or negative proper fraction ($-1<r<1$), the successive terms decrease in numerical value. The last case is of special interest.

As an example, consider the progression

$$1, \ \tfrac{1}{2}, \ \tfrac{1}{4}, \ \tfrac{1}{8}, \ \tfrac{1}{16}, \text{ etc.},$$

of which the general term is

$$T_n=\frac{1}{2^{n-1}}.$$

The sum of the first n terms is

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^n}\right).$$

S_n is always less than 2, and differs from 2 by $\frac{1}{2^{n-1}}$. This difference becomes less and less as more terms of the series are taken into the sum, S_n; that is, as n is taken increasingly larger; and it is evident that $\frac{1}{2^{n-1}}$ may be made less than any conceivable

number by making n large enough; that is, by taking enough terms of the series. Such a relation as this is briefly expressed by saying that

The limit of S_n as n increases indefinitely is 2.

A more usual expression is:

The limit of S_n when n is infinite is 2.

The formulation for this is:

The limit of S_n when $n=\infty$ is 2,

or more briefly,

$$S_n]_{n=\infty}=2.$$

This limit is called the *sum of the geometric progression to infinity,* or simply the *sum of the progression,* and is abbreviated S_∞ or S.

In general, in any geometric progression

$$a,\ ar,\ ar^2,\ \ldots.,\ ar^{n-1},\ \ldots.,$$

using the above notation, we have

$$S_n=a\frac{1-r^n}{1-r}=\frac{a}{1-r}(1-r^n).$$

And if r is less than unity in numerical value, a sufficiently large value of n will make r^n less than any conceivable number so that S_n gets nearer to the value $\frac{a}{1-r}$ as more and more terms are taken, and may be made to differ from $\frac{a}{1-r}$ by less than any conceivable value by taking enough terms; that is, by making n large enough:

$$S_n]_{n=\infty}=\frac{a}{1-r}(1-r^n)]_{n=\infty}=\frac{a}{1-r}(1-0)=\frac{a}{1-r},$$

or

$$S_\infty=\frac{a}{1-r},\quad \text{if}\quad -1<r<1.$$

For example, the sum to infinity of the geometric progression

1, 0.999, etc.,

for which $a=1, r=0.999$, is

$$S_\infty = \frac{1}{.001} = 1000.$$

The nearer r is to 1 in numerical value, the larger the sum of the progression; if r is equal to 1 or numerically greater, the value of S_n does not approach any limit as more terms are taken, but increases indefinitely, so that we say that the sum is infinite.

$$S_\infty = \infty, \quad \text{if} \quad r>1, r<-1, \text{or } r=1.$$

Examples.

1. Find the sum to infinity of the progressions

1, $\frac{1}{3}$, $\frac{1}{9}$, etc., and 1, $\frac{80}{81}$, $\frac{6400}{6561}$, etc.

Ans. $\frac{3}{2}$, 81.

2. How many terms must be taken into S_n in the case of each of the series in Example 1 before the difference $\frac{ar^n}{1-r}$ between S_n and S_∞ is less than 0.00001? [Use logarithms.]

Ans. 11 and 1280.

3. Find the sum to infinity of

6, -3, $\frac{3}{2}$, $-\frac{3}{4}$, etc. *Ans.* 4.

4. In a geometric progression $a=1$, $r=\frac{1}{2+\sqrt{3}}$; find S_∞.

Ans. $\frac{1+\sqrt{3}}{2}$.

5. Find the sums of the repetends $0.\dot{0}3\dot{7}$ and $0.\dot{0}\dot{9}$, treating them as geometric progressions. *Ans.* $\frac{1}{27}$ and $\frac{1}{11}$.

6. Reduce the repetend $2.31\dot{2}4\dot{3}$ to the equivalent fraction.

Ans. $\frac{2312}{925}$.

7. Two places, A and B, are a mile apart; a man walks from A toward B, going $\frac{3}{4}$ of the distance; turns, and walks $\frac{3}{4}$ of the way back toward A; turns, and walks $\frac{3}{4}$ of the way back toward his previous turning point. If he keeps this up indefinitely, how many miles will he walk, and how far will he get from A? *Ans.* 3 mi., $\frac{3}{7}$ mi.

8. A square is inscribed in a circle of radius a, a circle in it, a square in the second circle, and so on indefinitely. Find the sum to infinity of the perimeters and of the areas of all the squares. *Ans.* $8a(1+\sqrt{2})$, $4a^2$.

9. A cube is inscribed in a sphere of radius a, a sphere in it, a cube in the second sphere, and so on indefinitely. Find the sum to infinity of the surfaces of all the cubes. *Ans.* $12a^2$.

10. A regular tetraedron is inscribed in a sphere, a sphere in it, a regular tetraedron in the second sphere, and so on indefinitely. If the radius of the first sphere is a, find the sum to infinity of the volumes of all the spheres. [NOTE.—The altitudes of a regular tetraedron quadrisect one another.]

Ans. $\frac{1}{18}\pi a^3$.

43. Limits and Infinites.—The use of limits made in the preceding article occurs in many problems, as does the use, always in the same sense as there, of the words *infinite* and *infinity*. In such work certain expressions are used for brevity, the meaning of which it is necessary to recognize clearly.

A variable which decreases in such a way that by continuing in its variation it will become less in numerical value than any given number (however small) is said to *decrease indefinitely* or to *approach the limit zero*. Such a variable is called an *infinitesimal*.

A variable which increases in such a way that by continuing in its variation it will become greater in numerical value than any given number (however large) is said to *increase indefinitely*, to be *infinite* or to *equal infinity*. The symbol used for the word "infinity" is ∞; for the phrase "increase indefinitely," is $= \infty$.

A constant number, neither zero nor infinite, is called in distinction a *finite* number.

When a function, $f(x)$, owing to some variation given to its independent variable x, approaches closer and closer to a number L in such a way that the difference $L-f(x)$ decreases indefinitely, $f(x)$ is said to approach L as a limit.

If there are two functions, $f(x)$ and $F(x)$, it is evident that:

I. If $f(x) \equiv F(x)$, and if one of these forms can be shown to approach a limit, the other must approach the same limit.

II. If $f(x) < a$, and $F(x) > a$, a being some given number, and if $F(x) - f(x)$ decreases indefinitely, $f(x)$ and $F(x)$ must each approach a as a limit.

III. If $f(x)$ approaches a limit l and $F(x)$ approaches a limit L, $f(x) \pm F(x)$, $f(x) \times F(x)$, and $\frac{f(x)}{F(x)}$ approach the limits $l \pm L$, $l \times L$, and $\frac{l}{L}$, respectively.

As limits behave in all these ways like particular values of their function, the same notation is used for the limit of a function of x when x approaches a given limit as for the value of the function when x has a given value.

For instance, as x increases indefinitely, $\frac{1}{x}$ evidently decreases indefinitely, and vice versa; these relations are expressed either as follows:

$$\frac{1}{x}\Big]_{x=\infty} = 0 \quad \text{and} \quad \frac{1}{x}\Big]_{x=0} = \infty;$$

or as follows: If $f(x) \equiv \frac{1}{x}$,

$$f(\infty) = 0 \quad \text{and} \quad f(0) = \infty.$$

A still briefer expression for these same relations is:

$$\frac{1}{\infty} = 0, \quad \text{and} \quad \frac{1}{0} = \infty.$$

Examples.

1. In the study of the infinite geometric progression, what variable was an infinitesimal, what an infinite, and what approached a finite limit?

2. Give the values of the following limits:

$$\frac{3x^2+2}{x-3}\Big]_{x=0},\quad \sqrt{x^2-x+4}\Big]_{x=0},\quad \frac{1}{x}+\frac{1}{x-3}\Big]_{x=\infty}.\quad \textit{Ans.}\ -\tfrac{2}{3}, 2, 0.$$

3. Give the values of the following limits:

$$x^2]_{x=0},\quad \sqrt{x}]_{x=0},\quad x^2-\sqrt{x}]_{x=0},\quad (x-1)^2+\sqrt{x-1}]_{x=1},$$
$$(x+1)\sqrt{x^2-1}]_{x=1}.\quad \textit{Ans.}\ 0 \text{ for each.}$$

4. If two functions are both approaching the limit zero, what limit is approached by their sum? By their difference? By their product?

5. Give the values of the following limits:

$$x+2]_{x=\infty},\quad \frac{1}{\sqrt{x-1}}\Big]_{x=1},\quad \frac{1}{\sqrt{x-1}}+\frac{1}{\sqrt{x^2-1}}\Big]_{x=1},\quad \frac{1}{\sqrt{x+2}}$$
$$\times\frac{1}{\sqrt{x^2-4}}\Big]_{x=-2},\quad x\sqrt{x+2}]_{x=\infty}.\quad \textit{Ans.}\ \infty \text{ for each.}$$

6. If each of two functions is increasing indefinitely, how is their sum changing? Their product?

7. Give the values of the following limits:

$$x+\frac{1}{x}\Big]_{x=0},\quad x+\frac{1}{x}\Big]_{x=\infty},\quad \frac{x^2+2}{x-1}\Big]_{x=1},\quad \frac{x-1}{x^2+2}\Big]_{x=1}.$$
$$\textit{Ans.}\quad \infty,\ \infty,\ \infty,\ 0.$$

44. Operations with Infinitesimals and Infinites.—It is evident that if two variables are each approaching the limit zero, their sum, their difference, and their product also approach the limit zero; in other words, the sum, difference or product of two infinitesimals is an infinitesimal. Again, if two variables are each increasing indefinitely, their sum and product are also increasing indefinitely; or, the sum or product of two infinites is infinite. These relations are abbreviated or formulated:

$$0\pm 0=0,\quad 0\times 0=0,\quad \infty+\infty=\infty,\quad \infty\times\infty=\infty.$$

Many propositions can be written in similar abbreviated forms, as for instance $\frac{a}{0}=\infty$. This means that if the numerator of a fraction is constant or is approaching a finite limit, a, and

its denominator is approaching the limit zero, the value of the fraction is increasing indefinitely.

Exercises.

Explain in detail the meaning of $\frac{a}{\infty}=0, \frac{0}{\infty}=0, \frac{\infty}{0}=\infty$.

45. Indeterminate Forms.—Unlike the preceding cases, the quotient of two infinitesimals or of two infinites cannot be determined from the limits alone, but depends on the nature of the functions themselves; for in a fraction whose numerator and denominator are either both decreasing indefinitely or both increasing indefinitely, the variation of one term tends to increase the fraction, while the variation of the other tends to decrease it. The former influence may prevail, and the fraction be infinite, or the latter may, causing the fraction to approach zero as a limit, or the two may so balance each other that the fraction will approach some finite limit.

The two forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are therefore said to be indeterminate.

The following examples are typical of the treatment of these forms when the terms of the fraction are rational:

I. $$\frac{x^2+x-6}{x^2-6x+8}\Big]_{x=2}=\frac{0}{0},$$

but

$$\frac{x^2+x-6}{x^2-6x+8}\equiv\frac{(x-2)(x+3)}{(x-2)(x-4)}\Big]_{x=2}=\frac{x+3}{x-4}\Big]_{x=2}=-\frac{5}{2}.$$

II. $$\frac{3n^2+2n-1}{2n^2-n+3}\Big]_{n=\infty}=\frac{\infty}{\infty},$$

but

$$\frac{3n^2+2n-1}{2n^2-n+3}\equiv\frac{3+\frac{2}{n}-\frac{1}{n^2}}{2-\frac{1}{n}+\frac{3}{n^2}}\Big]_{n=\infty}=\frac{3}{2}.$$

The product of an infinitesimal and an infinite $(0\times\infty)$ can always be reduced to an indeterminate fraction, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the difference between two infinites $(\infty-\infty)$ to the form $\frac{\infty}{\infty}$.

With irrational and transcendental expressions, special devices or theories may be needed.

Examples.

1. Show that $(x-2)^2\times\frac{x+3}{x^2-6x+8}\Big]_{x=2}=0$ and that $\frac{x^2+x-6}{(x-2)(x^2-6x+8)}\Big]_{x=2}=\infty$.

2. Show that $\frac{3n^3+2n-1}{2n^2-n+4}\Big]_{n=\infty}=\infty$ and that $\frac{3n^2+2n-1}{2n^3-n+4}\Big]_{n=\infty}=0$.

3. Evaluate $\frac{(x^2-1)+(x-1)^2}{(x^3-1)-(x-1)^3}\Big]_{x=0}$, $\frac{(x^2-1)+(x-1)^2}{(x^2+2)-2(x+1)^2}\Big]_{x=0}$.
Ans. $\frac{2}{3}$, $\frac{1}{2}$.

4. Evaluate $\frac{(x^2-1)+(x-1)^2}{(x^3-1)-(x-1)^3}\Big]_{x=1}$, $\frac{(x^2-1)+(x-1)^2}{(x^2+7)-2(x+1)^2}\Big]_{x=1}$.
Ans. $\frac{2}{3}$, $-\frac{1}{3}$.

5. Evaluate $\frac{n^3-3n^2+2n-5}{6n^3-1}\Big]_{n=\infty}$, $\frac{2x^2-7x+3}{3x^2+5x-2}\Big]_{x=\infty}$.
Ans. $\frac{1}{6}$, $\frac{2}{3}$.

6. If $f(x)\equiv\frac{(x-1)(x^2-5x+6)}{(x-3)(2x^2+5x+2)}$, find $f(-2)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(\infty)$. *Ans.* ∞, 1, 0, 0, $\frac{2}{35}$, $\frac{1}{2}$.

7. Show that $\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\Big]_{x=0}=-1$, $\frac{1}{\sqrt{1-x}}-\frac{1}{\sqrt{1-x^2}}\Big]_{x=1}=\infty$.
[Rationalize the numerator in the first.]

8. What are $\log x]_{x=0}$, $\log x]_{x=\infty}$?

9. Let OA and OB be two perpendicular radii of a circle, and P a variable point of the circumference between A and B. Drop a perpendicular PM to OA, and write the six trigonometric functions of the angle POA. Show what limits they approach: first, as P approaches A; second, as P approaches B.

Ans. The results may be tabulated:

	sin	*cos*	*tan*	*cot*	*sec*	*csc*
0°:	0	1	0	∞	1	∞
90°:	1	0	∞	0	∞	1

46. Limiting Cases of Roots of Quadratic Equations.—We have seen that the roots of a quadratic equation

$$ax^2+bx+c=0$$

are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

We can write each of the roots in two equivalent forms by rationalizing the numerator:

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-2c}{+b+\sqrt{b^2-4ac}},$$

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-2c}{+b-\sqrt{b^2-4ac}}.$$

We will investigate the changes occurring in these roots when one (or more) of the coefficients decreases toward the limit zero.

Evidently (I) if c approaches zero, one of the roots approaches $-\frac{b}{a}$, the other approaches zero; and (II) if b and c both approach zero, both roots approach zero.

III. Let a approach zero. Then

$$\left[x_1=\frac{-2c}{b+\sqrt{b^2-4ac}}\right]_{a=0}=\frac{-2c}{2b}=-\frac{c}{b},$$

$$\left[x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}\right]_{a=0}=\frac{-2b}{0}=\infty.$$

That is, one root approaches the finite limit $\frac{-c}{b}$, the other increases indefinitely.

IV. Let a and b both approach zero. Then

$$\left[x_1=\frac{-2c}{b+\sqrt{b^2-4ac}}\right]_{a=b=0}=\frac{-2c}{0}=\infty,$$

$$\left[x_2=\frac{-2c}{b-\sqrt{b^2-4ac}}\right]_{a=b=0}=\frac{-2c}{0}=\infty.$$

That is, both roots increase indefinitely.

Applications of III occur in the simultaneous solution of a linear equation and a quadratic, if it happens that on eliminating one of the variables we obtain an equation of the first degree. In such a case we understand that of the two roots ordinarily to be expected one has disappeared through becoming infinite; that is, a slight change in the coefficients of the original equation would cause this missing root to appear as a very large quantity.

If the equation that results from the elimination is in the form of a constant equated to zero, IV shows that both of the roots ordinarily to be expected have become infinite.

The principles III and IV can be deduced in another way.

Dividing the original equation by x^2, we have:

$$c\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+a=0.$$

Here, if $a=0$, we get

$$c\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)=0,$$

whence

$$\frac{1}{x}=-\frac{b}{c}\text{ or }0,\quad x=-\frac{c}{b}\text{ or }\infty.$$

Again, if $a=b=0$, we get

$$c\left(\frac{1}{x}\right)^2=0,$$

whence

$$\frac{1}{x}=0\text{ twice},\quad x=\infty\text{ twice}.$$

Examples.

The distances of a point P from two given perpendicular straight lines are x and y respectively. Find these distances from the relations given in each of the examples below:

1. $3x^2-xy+y^2+6x+6y=0$, $y=2x$ $\Big\}$ $x=0,\ y=0$ or $x=-\frac{18}{5},\ y=-\frac{36}{5}$.
2. $3x^2-xy+y^2+6x+6y=0$, $y=-x$ $\Big\}$ $x=0,\ y=0$ twice.
3. $2x^2-3xy-2y^2+3=0$, $x=2y+1$ $\Big\}$ $x=-1,\ y=-1$ or $x=\infty,\ y=\infty$.
4. $2x^2-3xy-2y^2+3=0$, $x=2y$ $\Big\}$ $x=\infty,\ y=\infty$ twice.

47. Permutations and Combinations.—The different orders in which a number of things can be arranged are called their *permutations.*

For instance, the permutations of the three letters *a, b, c* are:

abc, acb, bac, bca, cab, cba.

The different arrangements of the same letters taken two at a time are:

ab, ba, ac, ca, bc, cb.

Each distinct group which may be selected from a number of things, regardless of their arrangement, is called a *combination.*

Thus, of the six permutations of the three things *a, b, c* taken all together, there is but one selection or combination of three different things, *abc.* The number of combinations of the same things taken two at a time is only three, *ab, ac,* and *bc.*

48. Number of Permutations of n Different Things.—This is the same as the number of ways in which n things may be arranged in a row of n places.

We may fill the first place in the row with any one of the n things; that is, in n different ways.

In each case we may fill up the second place in $(n-1)$ ways, i. e., with any one of the $(n-1)$ things left, making $n(n-1)$

different ways in which the two places may be filled, since we may follow any one of the n ways by any one of the $(n-1)$ ways. Similarly we may fill the third place in any one of $(n-2)$ ways, and the three places in $n(n-1)(n-2)$ ways. Finally, we can fill the nth place in only one way, as we shall have used the $(n-1)$ things in filling the $(n-1)$ preceding places. Hence, *there are*

$$n(n-1)(n-2)\ldots.3\cdot 2\cdot 1=\underline{|n}$$

different permutations of n things taken all together.

This is formulated

$$_nP_n=\underline{|n},$$

the symbol $\underline{|n}$ being called "*factorial n.*"

49. Permutations of *n* Different Things Taken *s* at a Time.—Counting the number of permutations of n things taken s at a time is the same as finding in how many different ways we may fill up a row of s places with n things; the number is represented by

$$_nP_s.$$

We may fill up the first place in n ways, the first and second in $n(n-1)$ ways, the first three places in $n(n-1)(n-2)$ ways, just as in the preceding article, until, when the last place, number s, is filled, we shall have

$$n(n-1)(n-2)\ldots.(n-s+1),$$

or

$$_nP_s=n(n-1)(n-2)\ldots.(n-s+1).$$

If we multiply and divide this expression by the remaining $(n-s)$ factors, it reduces to

$$_nP_s=\frac{\underline{|n}}{\underline{|n-s}},$$

as the formula for finding the number of permutations of n things taken s at a time.

Examples.

1. In how many ways can a crew of 8 men be seated in an eight-oared shell? *Ans.* 40320.

2. How many six-letter arrangements can be made of the letters *causing*? *Ans.* 5040.

3. How many four-letter arrangements can be made from the letters *causing*? *Ans.* 840.

4. How many four-figure numbers can be made out of the digits 1, 2, 3, 4, 5 without repetitions? How many five-figure numbers? How many numbers? *Ans.* 120, 120, 325.

5. How many of the integers between 50,000 and 60,000 can be formed from the digits 1, 3, 5, 7, 9 used without repetitions? *Ans.* 24.

6. How many permutations of 9 letters each can be made from 9 different letters if 3 of the letters must always come together in a given order? *Ans.* 5040.

7. How many permutations of 6 letters each can be made from 9 different letters, of which 3, if they are taken at all, must occur together and in a fixed order? *Ans.* 1200.

COMBINATIONS.

50. Number of Combinations of n Different Things Taken s at a Time.—Suppose there are x combinations each containing s things; from each combination there can be made $\lfloor s$ permutations, and from the x combinations $x\lfloor s$ permutations, which will be all of the possible $\frac{\lfloor n}{\lfloor n-s}$ permutations. Hence

$$x\lfloor s = \frac{\lfloor n}{\lfloor n-s}; \text{ and } x = \frac{\lfloor n}{\lfloor s \lfloor n-s}.$$

This is formulated,

$${}_nC_s = \frac{\lfloor n}{\lfloor s \lfloor n-s}.$$

51. If we find from this formula the number of combinations of n things taken $(n-s)$ at a time, we obtain

$${}_nC_{n-s} = \frac{\lfloor n}{\lfloor n-s \lfloor n-(n-s)} = \frac{\lfloor n}{\lfloor s \lfloor n-s},$$

or

$${}_nC_s = {}_nC_{n-s}.$$

That is, *the number of combinations of n things taken* $(n-s)$ *at a time is the same as the number of combinations of n things taken s at a time.*

Examples.

1. In how many ways may a guard of 5 men be selected from a troop of 20 men? *Ans.* 15504.
2. How many different eight-oared crews can be selected from 16 men? *Ans.* 12,870, positions undistinguished.
3. How many different parties of 5 each can be selected from 15 men? *Ans.* 3003.
4. If ${}_{n+2}C_4 = 11\,{}_nC_2$, find n and check.
5. If ${}_nC_s = 3\,{}_nC_{s-1}$, find n in terms of s and check.
6. If $5\,{}_nC_6 = 12\,{}_nC_4$, find n and check. *Ans.* $n=13$.
7. If $5\,{}_9C_s = 18\,{}_7C_s$, find s and check. *Ans.* $s=4$.

52. An Important Principle.—It is important to note carefully the principle used several times already, that if one thing can be done in a ways, and, without interfering with the first performance, another thing can be done in b ways, both things can be done in ab ways.

For instance, if a troop of 50 men and a captain are to be chosen from 55 men and 5 officers, since the men can be chosen in ${}_{55}C_{50}$ ways, and the captain in 5, the troop can be made up in $5 \times {}_{55}C_{50} = 17{,}393{,}805$ ways.

Examples.

1. In how many ways may 5 articles be put into 3 boxes, each capable of holding all 5 articles? *Ans.* 243.
2. How many signals may be made with 10 numeral flags, each signal to consist of from 1 to 3 flags, repetitions being allowed? *Ans.* 1110.
3. In how many ways can the crew of an eight-oared shell be seated if 4 of them can row only on the port side? *Ans.* 576.
4. A pitcher throws 4 different curves and a straight ball; in how many ways can he pitch 4 times? In how many if he throws differently each time? *Ans.* 625, 120.
5. How many baseball teams can be formed from 15 players, of whom 2 can pitch only and 2 can catch only? *Ans.* 1320.

6. How many four-letter arrangements can be made using the letters *causing* without repetitions, if consonants and vowels alternate? *Ans.* 144.

7. Of a crew of 8 men, 3 can row only on the port side, 2 only on the starboard, the rest on either. How many ways are there of seating the crew? *Ans.* 1728.

53. Product of Linear Factors.—In order to find the rule by which we can write down without multiplication the product of any number of factors $(x+a)$, $(x+b)$, $(x+c)$, $(x+d)$, etc., we will first perform the actual multiplication for a few factors:

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ \quad bx+ab \\ \hline x^2+(a+b)x+ab \\ \quad x+c \\ \hline x^3+(a+b)x^2+abx \\ \quad cx^2+(ac+bc)x+abc \\ \hline x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \end{array}$$

Multiplying this result by $(x+d)$, we get

$$x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2 + (bcd+acd+abd+abc)x+abcd,$$

and so on.

It soon becomes evident, as we continue to multiply new factors into the product, that the following general rule is true:

The product of n factors $(x+a)$, $(x+b)$, $(x+c)$, $(x+d)$, etc., is a polynomial of degree n, consisting of $(n+1)$ terms, in which the coefficient of the various powers of x are as follows:

Of x^n : *Unity;*

Of x^{n-1}: *The sum of all the numbers a, b, c, d,* etc.;

Of x^{n-2}: *The sum of the products of the numbers a, b, c, d,* etc., taken two at a time;

Of x^{n-3}: *The sum of the products of the numbers a, b, c, d,* etc., taken three at a time;

Of x^{n-4}: *The sum of the products of the numbers a, b, c, d,* etc., taken four at a time;

and in general:

The coefficient of x^{n-k} is the sum of the products of the numbers a, b, c, d, etc., taken *k* at a time.

The last term comes under this general formula, being the coefficient of 1, or of x^{n-n}, and equal to the product of all n of the numbers $a, b, c, d,$ etc.

54. The Binomial Theorem.—If in the preceding theorem, the numbers $a, b, c, d,$ etc., are all equal, the product of the n factors becomes $(x+a)^n$; each product of two of the numbers $a, b, c, d,$ etc., becomes a^2; each product of three becomes a^3, and so on; each product of k of the numbers becomes a^k.

There are

$${}_nC_2 = \frac{\lfloor n}{\lfloor 2 \, \lfloor n-2}$$ products of two numbers chosen out of n,

$${}_nC_3 = \frac{\lfloor n}{\lfloor 3 \, \lfloor n-3}$$ products of three numbers chosen out of n,

$${}_nC_4 = \frac{\lfloor n}{\lfloor 4 \, \lfloor n-4}$$ products of four numbers chosen out of n,

and so on; finally, there are in general

$${}_nC_k = \frac{\lfloor n}{\lfloor k \, \lfloor n-k}$$ products of k numbers chosen out of n.

Hence

$$(x+a)^n \equiv x^n + nax^{n-1} + \frac{\lfloor n}{\lfloor 2 \, \lfloor n-2} a^2x^{n-2} + \frac{\lfloor n}{\lfloor 3 \, \lfloor n-3} a^3x^{n-3} + \ldots.$$

$$+ \frac{\lfloor n}{\lfloor k \, \lfloor n-k} a^kx^{n-k} + \ldots. + \frac{\lfloor n}{\lfloor n-2 \, \lfloor 2} a^{n-2}x^2$$

$$+ \frac{\lfloor n}{\lfloor n-1 \, \lfloor 1} a^{n-1}x + a^n.$$

Since ${}_nC_k = {}_nC_{n-k}$, the coefficients at equal distances from the two ends of the expansion are alike. The expansion with coefficients simplified is:

$$(x+a)^n \equiv x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} a^3x^{n-3}$$
$$+ \ldots + \frac{n(n-1)(n-2)\ldots(n-k+1)}{\lfloor k} a^kx^{n-k}$$
$$+ \ldots + \frac{n(n-1)(n-2)}{\lfloor 3} a^{n-3}x^3 + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2$$
$$+ na^{n-1}x + a^n.$$

Note that the general term of $(x+a)^n$,

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{\lfloor k} a^kx^{n-k},$$

is *the* $(k+1)$*-th term.*

This expression for the general term can be applied to finding any required term of a particular expansion; e. g., to find the fourth term of $(x+a)^{12}$. Here $k=3$, $n=12$; and the required fourth term is

$$\frac{n(n-1)(n-2)}{\lfloor 3} a^kx^{n-k} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} a^3x^9 = 220a^3x^9.$$

When it is desired to write out a complete expansion, it may be noticed that any term of $(x+a)^n$ may be got by multiplying the preceding term by its exponent of x, dividing by one more than its exponent of a, lowering the exponent of x one unit and raising the exponent of a one unit. For instance,

$$(x+a)^7 = x^7 + 7ax^6 + \frac{7 \cdot 6}{2} a^2x^5 + \frac{5 \cdot 21}{3} a^3x^4 + \frac{4 \cdot 35}{4} a^4x^3 + \ldots,$$

so that the coefficients are

1, 7, 21, 35, 35, 21, 7, 1.

The expansion of $(x-a)^n$ is given by the same theorem, since $(x-a)^n \equiv (x+(-a))^n$; the only difference is in the signs, which are alternately $+$ and $-$.

The letters x and a may of course stand for any values whatever.

Examples.

1. Write the first five terms of the expansion of each of the following: $(a+x)^n$, $(a-x)^n$, $(x-a)^n$, $(a+x)^9$.

2. Write the full expansion of each of the following:
$$(x+a)^5,\ (x-a)^3,\ (a-x)^6,\ (a+x)^4.$$

3. Write the full expansions of the following:
$$(x+2)^4,\ \left(1-\frac{x}{2}\right)^5,\ (\tfrac{2}{3}x+\tfrac{3}{2}a)^6,\ (1+\tfrac{2}{3}x)^4.$$

4. Find the seventh term of $(a-x)^{10}$.

5. Find the fourth term of $\left(x-\frac{2a}{3}\right)^{20}$. *Ans.* $\frac{-3040}{9}a^3x^{17}$.

6. Find the term containing x^2 in $\left(\frac{\sqrt{x}}{a}-\frac{a}{\sqrt[3]{x}}\right)^{14}$.

Ans. $3003\frac{x^2}{a^2}$.

55. The Binomial Theorem for Fractional and Negative Exponents.—It is proved by the Calculus that the use of the Binomial Theorem can be extended to fractional and negative powers of a binomial under certain restrictions.

$$(a+x)^n \equiv a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ldots.$$

gives an unending succession of terms if n is fractional or negative; but if x is numerically less than a (and only then), the sum formed by adding more and more of these terms will approach as a limit the value of $(a+x)^n$.

In such an application of the Binomial Theorem, it is convenient to use the following expansion, which holds true for any value of x numerically less than 1:

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4+\ldots..$$

For the sake of compactness and accuracy, the work is best arranged in the form shown in the following example:

Expand $(4-x)^{-\frac{1}{2}}$ to five terms.

$$(4-x)^{-\frac{1}{2}} \equiv 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \equiv \frac{1}{8}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}.$$

1	1	1	1	1
	$-\frac{3}{2}$	$\frac{15}{8}$	$-\frac{35}{16}$	$\frac{315}{128}$
	$-\frac{x}{4}$	$\frac{x^2}{4^2}$	$-\frac{x^3}{4^3}$	$\frac{x^4}{4^4}$

$$(4-x)^{-\frac{1}{2}} \equiv \frac{1}{8}\left(1+\frac{3}{2}\cdot\frac{x}{4}+\frac{15}{8}\cdot\frac{x^2}{4^2}+\frac{35}{16}\cdot\frac{x^3}{4^3}+\frac{315}{128}\cdot\frac{x^4}{4^4}\right)$$

The first line contains the powers of 1; the second line contains the coefficients, each of which is derived from the preceding one by multiplying by an appropriate factor; the third line contains the powers of $\left(-\frac{x}{4}\right)$. The three parts of each term lying in the same vertical column are multiplied together.

Note especially the simple method of getting the coefficients; the first is n; multiplied by $\frac{n-1}{2}$, it gives the second; this product multiplied in turn by $\frac{n-2}{3}$ gives the third; this product times $\frac{n-3}{4}$ gives the fourth, and so on; that is, the first coefficient is $-\frac{3}{2}$; $-\frac{3}{2}\left(-\frac{5}{2\cdot 2}\right)=\frac{15}{8}$; $\frac{15}{8}\left(-\frac{7}{2\cdot 3}\right)=-\frac{35}{16}$; $-\frac{35}{16}\left(-\frac{9}{2\cdot 4}\right)=\frac{315}{128}$.

When such an expansion as this is applied to numerical computation, this use of multipliers can be carried a step further to advantage. We note that any complete term can be got by multiplying the preceding term by a simple factor; these multipliers are, for the expansion of $(1+x)^n$:

To get the second term from the first: nx,

To get the third term from the second: $\frac{n-1}{2}x$,

To get the fourth term from the third: $\frac{n-2}{3}x$,

and so on.

Example: Find to eight decimals the value of $\frac{1}{\sqrt{(3.9)^5}}$ $=(4-\frac{1}{10})^{-\frac{5}{2}}=\frac{1}{8}(1-\frac{1}{40})^{-\frac{5}{2}}$.

Using the notation T_1 for first term, T_2 for second term, T_3 for third term, etc., we have $T_1=1$, $T_2=nx=(-\frac{5}{2})(-\frac{1}{40})=\frac{5}{80}$, $T_3=\frac{n-1}{2}x\cdot T_2=\frac{\frac{5}{2}-1}{2}\left(-\frac{1}{40}\right)T_2=\frac{5}{4\cdot 40}T_2=\frac{1}{32}T_2$, etc. The whole computation is conveniently arranged as follows:

$T_1=$	1.000000000
$T_2=\quad nx=-\frac{5}{2}(-\frac{1}{40})=\frac{5}{80}=$	.037500000
$T_3=\frac{n-1}{2}xT_2=(-\frac{3}{4})(-\frac{1}{40})\,T_2=\frac{3}{32}T_2=$	.001171875
$T_4=\frac{n-2}{3}xT_3=(-\frac{7}{6})(-\frac{1}{40})\,T_3=\frac{7}{240}T_3=$	.000034180
$T_5=\frac{n-3}{4}xT_4=(-\frac{9}{8})(-\frac{1}{40})\,T_4=\frac{9}{320}T_4=$	.000000961
$T_6=\frac{n-4}{5}xT_5=(-\frac{11}{10})(-\frac{1}{40})\,T_5=\frac{11}{400}T_5=$	.000000026
$T_7=\frac{n-5}{6}xT_6=(-\frac{13}{12})(-\frac{1}{40})\,T_6=\frac{13}{480}T_6=$	.000000001
	8)1.038707043
	.12983838

The work has been carried to nine decimals, that the eighth place may be correct.

In an example of this sort it becomes evident after a few multipliers have been computed that the others will follow according to some simple rule; here the coefficients of $\frac{1}{40}$ in the multipliers have successive odd numbers for numerators and successive even numbers for denominators.

Examples I.

Expand each of the following to five terms:

1. $\frac{1}{1+x}$. *Ans.* $1-x+x^2-x^3+x^4$.
2. $\frac{1}{\sqrt{1+x}}$. *Ans.* $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4$.
3. $\sqrt{1+x}$. *Ans.* $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$.
4. $\sqrt{1-x^2}$. *Ans.* $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8$.

5. $\frac{1-x}{1+x}=1-2x\cdot\frac{1}{1+x}$. *Ans.* $1-2[x-x^2+x^3-x^4]$.

6. $(1-x)^{\frac{1}{2}}$. *Ans.* $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{7}{128}x^4$.

7. $(1+x)^{-\frac{1}{3}}$. *Ans.* $1-\frac{1}{3}x+\frac{2}{9}\frac{1}{2}x^2-\frac{7}{1}\frac{7}{2}\frac{7}{8}x^3+\frac{1}{2}\frac{1}{0}\frac{5}{4}\frac{5}{8}x^4$.

Examples II.

Find the value of each of the following to the number of decimal places indicated, carrying the work in each case one place further than is required in the result.

1. $\frac{1}{1.03}$ (eight places). *Ans.* 0.97087379.

2. $\frac{1}{\sqrt{1.1}}$ (seven places). *Ans.* 0.9534626.

3. $\frac{7}{5}\sqrt{1+\frac{1}{49}}=\sqrt{2}$ (seven places). *Ans.* 1.4142136.

4. $\frac{7}{4}\sqrt{1-\frac{1}{49}}=\sqrt{3}$ (seven places). *Ans.* 1.7320508.

5. $\sqrt[5]{34^2}$ (eight places). *Ans.* 4.09818507.

6. $\sqrt[3]{2}=\frac{5}{4}\sqrt[3]{1+\frac{8}{125}}$ (ten places). *Ans.* 1.2599210499.

7. $\sqrt{2}=\frac{10}{7}\sqrt{1-\frac{1}{50}}$ (seven places).

8. $\frac{1}{\sqrt[7]{(130)^3}}=\frac{1}{5}(1+\frac{1}{64})^{-\frac{3}{7}}$ (seven places). *Ans.* 0.1241722.

56. The Expansion of $(a+bi)^n$.—In the expansion of $(a+bi)^n$, where a and b are real and $i=\sqrt{-1}$, we have only to note that $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, etc., the powers repeating in cycles of four. As the first term contains i^0, the third i^2 and so on, every odd-numbered term is real, with signs alternately $+$ and $-$; the even-numbered terms in the same way contain odd powers of i, equal to i, $-i$, i, $-i$ alternately. Hence, collecting separately the real and imaginary parts of the expansion:

$$(a+bi)^n\equiv a^n-\frac{n(n-1)}{\lfloor 2}a^{n-2}b^2+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}a^{n-4}b^4-\ldots.$$

$$+i\left[na^{n-1}b-\frac{n(n-1)(n-2)}{\lfloor 3}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}a^{n-5}b^5-\ldots.\right].$$

Thus $(a+bi)^n\equiv P+Qi$, where P and Q are real.

In the expansion of $(a-bi)^n$, the only difference is that b of

$(a+bi)^n$ is replaced by $-b$. This change does not affect P, as only even powers of b occur in P, and merely changes the sign of Q, as only odd powers occur in Q; hence

$$(a+bi)^n \equiv P+Qi,$$

and

$$(a-bi)^n \equiv P-Qi,$$

where P and Q are real.

From this it follows that if we have any polynomial, $f(x)$, with real coefficients, and substitute in it $(a+bi)$ and $(a-bi)$ in turn, after collecting the real and pure imaginary parts we shall have, if $f(a+bi)=p+qi$, $f(a-bi)=p-qi$, the two results being conjugate.

If $f(x)$ is a polynomial with real coefficients,

$$f(a+bi)=p+qi,\ f(a-bi)=p-qi,$$

where p and q are real.

57. The Expansion of $(a+b\sqrt{c})^n$.—If a and b are rational, and $\sqrt{c}$ is irrational, odd powers of $\sqrt{c}$ are irrational, and even powers rational, so that a proof similar to that of the preceding article shows that

$$(a+b\sqrt{c})^n = P+Q\sqrt{c},$$

where P and Q are rational. The effect of changing b to $-b$ is the same as in the preceding case, so that

$$(a-b\sqrt{c})^n = P-Q\sqrt{c}.$$

Also, if $f(x)$ is a polynomial with *rational* coefficients, and the conjugate quadratic surds $(a+b\sqrt{c})$ and $(a-b\sqrt{c})$ are put in place of x, the results, $f(a+b\sqrt{c})$ and $f(a-b\sqrt{c})$, will be conjugate quadratic surds.

58. The Remainder Theorem.—Consider the cubic function

$$f(x) \equiv 2x^3+x^2-x+10.$$

If we divide it by $(x-2)$, the quotient will be $2x^2+5x+9$ and the remainder 28. Again, if we substitute 2 for x, we find $f(2)=28$. Dividing by $(x-1)$, we get 12 for a remainder, and substituting 1 for x we get $f(1)=12$. We may try this for any number, and find always that dividing $f(x)$ by $(x-a)$ will leave $f(a)$ for a remainder. This happens as follows: The first division shows that

$$f(x) \equiv 2x^3+x^2-x+10 \equiv (x-2)(2x^2+5x+9)+28.$$

If we put 2 for x throughout, the identity gives

$$f(2)=28.$$

In the general case, let the division of $f(x)$ by $(x-a)$ be carried out, as it always can be, until a remainder occurs not involving x; i. e., a constant number. Call this remainder R; then

$$f(x) \equiv (x-a)Q(x)+R,$$

$Q(x)$ being the quotient. This relation, true for any value of x, gives for the value a:

$$f(a)=(a-a)Q(a)+R=R.$$

It is evident that $f(x)$ in this proof may stand for any polynomial, so that we have the theorem:

If any polynomial, $f(x)$, be divided by $(x-a)$, the remainder found will be $f(a)$.

It therefore follows that:

If for any polynomial, $f(x)$, $f(a)=0$, then $(x-a)$ is a factor of $f(x)$; or *if a is a root of any algebraic equation, $f(x)=0$, then $(x-a)$ is a factor of its left member*; and conversely:

If $(x-a)$ is a factor of any polynomial, $f(x)$, then $f(a)=0$; or *if $(x-a)$ is a factor of the left member of any algebraic equation, then a is a root of the equation.*

EQUATIONS.

59. Reducing Equations; Factoring Polynomials.—The process of dividing an equation $f(x)=0$ by $(x-a)$, a being a root of the equation, is called *dividing out or removing the root a;* and the equation obtained, one unit lower in degree, and containing one root fewer, is called the *reduced equation.*

In the example of the preceding article, dividing the root (-2) out of the equation $2x^3+x^2-x+10=0$, we get the reduced equation $2x^2-3x+5=0$. It is evident that the roots of the reduced equation are roots of the original equation. We are thus enabled in this case to solve the cubic equation completely, its roots being -2, $\frac{1}{4}(3\pm\sqrt{-31})$.

Whenever the left member of an equation can be factored, the solution of the equation can be simplified.

Again, since we know that $\frac{1}{4}(3+\sqrt{-31})$ and $\frac{1}{4}(3-\sqrt{-31})$ are roots of $2x^2-3x+5=0$, we know that $2x^2-3x+5$ can be divided exactly by $x-\frac{1}{4}(3+\sqrt{-31})$ and by $x-\frac{1}{4}(3-\sqrt{-31})$; hence its factors are $[x-\frac{1}{4}(3+\sqrt{-31})][x-\frac{1}{4}(3-\sqrt{-31})]$ and some number not involving x, which evidently is 2. Finally we are able to factor the cubic function:

$$2x^3+x^2-x+10\equiv 2(x+2)[x-\tfrac{1}{4}(3+\sqrt{-31})][x-\tfrac{1}{4}(3-\sqrt{-31})].$$

Whenever we can solve the equation formed by setting a given polynomial equal to zero, we can factor the polynomial.

60. Factors of a Quadratic.—It is often important to know the nature of the factors of a quadratic expression such as may be represented in general by ax^2+bx+c or by $ax^2+bxy+cy^2$. Since the factors of ax^2+bx+c are found by solving the equation $ax^2+bx+c=0$, they will be real when b^2-4ac is positive, imaginary when b^2-4ac is negative, and equal when $b^2-4ac=0$. The same discrimination also holds for the form with two variables, as may be seen from the equation $a\left(\frac{x}{y}\right)^2+b\left(\frac{x}{y}\right)+c=0$.

61. Forming Equations.—Suppose we wish to form an equation having the roots -1, 2 and 3. Evidently the equation

$$(x+1)(x-2)(x-3)=0$$

has these roots, for if we make $x=-1$, or 2 or 3, one of the factors of the left member becomes zero, and the equation is satisfied. Any other equation having the same roots must be divisible by $(x+1)$, $(x-2)$ and $(x-3)$; if this other equation is only of degree 3, it will be $k(x+1)(x-2)(x-3)=0$, where k is some constant.

The general principles exemplified here are stated:

An equation having given roots a, b, c, etc., may be formed by setting equal to zero the product of the factors $(x-a)$, $(x-b)$, $(x-c)$, *etc.*

Two equations of the same degree having the same roots, have also the same coefficients, or can be made to have by dividing one of the equations by a constant factor.

An equation having n roots is of the n-th degree.

62. Imaginary and Irrational Roots.—Given an equation $f(x)=0$ of which the coefficients are real, and suppose that an imaginary, $a+bi$ (a and b real) is one of its roots; then

$$f(a+bi)=0.$$

But

$$f(a+bi)=P+Qi,$$

where P and Q are real; then

$$P+Qi=0,$$

so that $P=0$ and $Q=0$; consequently

$$f(a-bi)=P-Qi=0,$$

so that $a-bi$ is also a root.

Similarly, if the coefficients of $f(x)=0$ are rational, and a quadratic surd, $a+b\sqrt{c}$, is one of its roots, $a-b\sqrt{c}$ is also a root.

These facts are generally stated:

Imaginary and quadratic surd roots occur in conjugate pairs.

63. Number of Roots of an Algebraic Equation.—*Any algebraic equation of the n-th degree has just n roots; and any polynomial of the n-th degree can be factored (in only one way) into n linear factors.*

This follows immediately from the fact that any algebraic equation has at least one root, the proof of which cannot be given here. Assuming it, we see that we could reduce any given equation of the nth degree to one of degree $(n-1)$ by dividing out the assumed root; we could then treat the reduced equation in the same way, obtaining 2 roots in all of the original equation, and a reduced equation of degree $n-2$. Just n such steps are possible, so that there are n roots. The original equation can be factored in this way into n factors and written in the form

$$k(x-a)(x-b)(x-c)\ldots.=0.$$

There cannot be any root in addition to what would be found in this way, for if there were more than n roots, the degree of the equation would be higher than the nth. Furthermore, a different set of roots could not be found in some other way, for if we substitute for x a number p different from every one of the roots $a, b, c, \ldots.$, we get

$$k(p-a)(p-b)(p-c)\ldots.=0,$$

an untrue relation, since no one of the factors is zero.

Further, we have seen that if the coefficients of an algebraic equation are *real,* imaginary roots occur in pairs; hence:

An algebraic equation of odd degree, if its coefficients are real, has at least one real root.

Examples.

In each of the following, either $f(1)$ or $f(-1)$ is zero; aided by this information, factor $f(x)$ and give all the roots of $f(x)=0$.

1. $f(x) \equiv 3x^3 - 2x^2 + 4x - 5$.
2. $f(x) \equiv 2x^3 - 3x^2 - 7x - 2$.

Factor each of the following:

3. $6x^2 - 7x - 3$ and $x^2 - x + 1$.

Solve each of the following:

4. $(x^2 - 5x + 6)(2x^2 + 17x + 8) = 0$.
5. $x^3 - 11x^2 + 10x = 0$.

Form the rational equations of lowest degree having the roots indicated:

6. Roots $-\frac{3}{2}$, $\frac{2}{3}$, 1, -2.
7. Roots -1, 1, 4, 0.
8. Roots 100, 1, -1, -2.
9. One root is $3 - \sqrt{2}$, another -1.
10. Two of the roots are $1 - \sqrt{-3}$ and $1 - \sqrt{-2}$.
11. One root is $\sqrt{2} - \sqrt{3}$.

64. Relations of Roots and Coefficients.—From the two articles just preceding, it follows that if an algebraic equation is divided through by the coefficient of its term of highest degree, thus being reduced to the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots + p_n = 0,$$

its left member may be obtained by multiplying together n factors

$$(x - a)(x - b)(x - c)\ldots,$$

where a, b, c, etc., are the n roots.

According to the Theorem of the Product of Linear Factors (Art. 53):

$p_1 = -(a + b + c + \ldots)$ = the negative of the sum of the roots;

$p_2 = (ab + bc + ca + \ldots)$ = the sum of the products of the roots by twos;

$p_3 = -(abc + \ldots)$ = the negative of the sum of the products of the roots by threes;

and so on, until finally

p_n=the product of all the roots if n is even, the negative of this product if n is odd.

The chief use that we shall make of this theorem is in checking the results of computations of roots; for this purpose the two results

$$p_1 = -(a+b+c+\ldots.) \text{ and } p_n = \pm abc\ldots.$$

are used.

64a. Changing the Signs of all the Roots.—If, instead of a, b, c, etc., for the roots of the equation in the preceding article, we had used $-a$, $-b$, $-c$, etc., their negatives, the n factors would have been

$$(x+a)(x+b)(x+c)\ldots.,$$

with the same resulting coefficients except that none of the parentheses would be marked negative. Hence:

The signs of all the roots of an algebraic equation are changed by changing the signs of the coefficients of alternate powers of the variable.

Examples.

1. Write the equation whose roots are the negatives of the roots of $3x^3 - 2x^2 + x - 1 = 0$; of $4x^4 + 3x^2 + x - 5 = 0$; of $4x^4 - 3x + 5 = 0$; of $2x^3 - 2x + 1 = 0$.

2. Solve $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + \sqrt{-1}$.

3. Solve $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, one root being $2 - \sqrt{3}$.

Solve the following equations:

4. $2x^3 - 19x^2 + 57x - 54 = 0$, the roots forming a g. p.

5. $3x^3 - 16x^2 + 12x + 16 = 0$, one root being double one of the other roots.

6. $6x^4 - 7x^3 - 16x^2 + 21x - 6 = 0$, the product of two of the roots being -3.

7. $12x^4 - x^3 - 54x^2 + 4x + 24 = 0$, the sum of two roots being zero.

8. Show from the Law of Coefficients that if the area of a rectangle is a and its perimeter is p, its dimensions are the roots of $s^2 - \frac{p}{2}s + a = 0$.

CHAPTER IV.

Algebraic Processes.

65. Use of Detached Coefficients.—Consider the multiplication:

$$
\begin{array}{r}
7x^4+13x^3-\ 2x\ +\ 1 \\
3x^2-\ x\ +\ 3 \\
\hline
21x^6+39x^5\qquad -\ 6x^3+3x^2 \\
-\ 7x^5-13x^4\qquad +2x^3-\ x \\
21x^4+39x^3\qquad -6x+3 \\
\hline
21x^6+32x^5+\ 8x^4+33x^3+5x^2-7x+3
\end{array}
$$

It is not necessary to waste time and space by writing the powers of x; we may write only the coefficients, indicating the absence of missing powers by zeros, and letting the position of each coefficient tell what power of x it is meant to affect.

Thus abbreviated, the multiplication stands:

$$
\begin{array}{r}
7+13+\ 0-\ 2+1 \\
3-\ 1+3 \\
\hline
21+39+\ 0-\ 6+3 \\
-\ 7-13+\ 0+2-1 \\
21+39+0-6+3 \\
\hline
21+32+\ 8+33+5-7+3
\end{array}
$$

The degree of the product is the sum of the degrees of the factors, so that the product may be read off by inserting mentally the powers 6, 5, 4, 3, 2, 1, 0 of x after the coefficients.

The inverse process of division may be similarly simplified; e. g.:

$$
\begin{array}{l}
3-1+3)21+32+\ 8+33+5-7+3(7+13+0-2+1 \\
\quad 21-\ 7+21 \\
\quad\quad\overline{\quad\quad} \\
\quad\quad 39-13+33 \\
\quad\quad 39-13+39 \\
\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad 0-\ 6+5-7 \\
\quad\quad\quad\quad\ -\ 6+2-6 \\
\quad\quad\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad\quad\quad 3-1+3 \\
\quad\quad\quad\quad\quad\quad 3-1+3 \\
\quad\quad\quad\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad\quad\quad 0+0+0
\end{array}
$$

These processes are said to be performed with *detached coefficients.*

66. Synthetic Division by a Binomial.—Division by a binomial of the form $(x \pm a)$ can be further simplified. Consider the process:

$$(7x^4+13x^3+2x+1) \div (x-3).$$

$$
\begin{array}{l}
1-3)7+13+0+2+1(7+34+102+308 \\
\quad 7-21 \\
\quad\quad\overline{\quad\quad} \\
\quad\quad 34+\ 0 \\
\quad\quad 34-102 \\
\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad 102+\ 2 \\
\quad\quad\quad\quad 102-306 \\
\quad\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad\quad\quad 308+\ 1 \\
\quad\quad\quad\quad\quad\quad 308-924 \\
\quad\quad\quad\quad\quad\quad\overline{\quad\quad} \\
\quad\quad\quad\quad\quad\quad\quad\quad 925
\end{array}
$$

Quotient:

$$7x^3+34x^2+102x+308+\frac{925}{x-3}.$$

The numbers 7, 34, 102 and 308, as well as 0, 2, 1, the numbers "brought down," have been written twice, though once is enough. The essential figures are:

$$\begin{array}{l} -3)7+13+0+2+1(7+34+102+308 \\ \quad -21 \\ \quad \overline{} \\ \quad\quad 34-102 \\ \quad\quad \overline{} \\ \quad\quad\quad 102-306 \\ \quad\quad\quad \overline{} \\ \quad\quad\quad\quad 308-924 \\ \quad\quad\quad\quad \overline{} \\ \quad\quad\quad\quad\quad 925 \end{array}$$

Moving the double diagonal line into a horizontal position, for the sake of compactness, we have:

$$\begin{array}{l} -3)7+13+\ \ 0+\ \ 2+\ \ 1 \\ \quad\quad -21-102-306-924 \\ \quad\quad \overline{7+34+102+308+925} \end{array}$$

No separate writing of the quotient is necessary; the coefficients on the lowest line give quotient and remainder; the quotient is of degree 3 (one less than that of the dividend), so that the result may be read directly. Finally, as addition is simpler than subtraction, $+3$ is put in place of -3 in the divisor, and the columns are *added,* giving the process of *synthetic division,* as follows:

$$\begin{array}{l} 3)7+13+\ \ 0+\ \ 2+\ \ 1 \\ \quad\quad +21+102+306+924 \\ \quad\quad \overline{7+34+102+308+925} \end{array}$$

Result:

$$7x^3+34x^2+102x+308+\frac{925}{x-3}.$$

67. Synthetic Division by a Polynomial.—Required the quotient of

$$(x^6+5x^5-x^4-10x^3+31x^2-19x+12) \div (x^3+2x^2-3x+2).$$

1	$1+5-1-10$	$+31-19+12$
-2	$-2-6+8$	-10
3	$3+9$	$-12+15$
-2	-2	$-6+8-10$
	$1+3-4+5$	$3+4+2$

Result:

$$x^3+3x^2-4x+5+\frac{3x^2+4x+2}{x^3+2x^2-3x+2}.$$

The detached coefficients of the dividend are written on a horizontal line; the detached coefficients of the divisor, with the signs of all but the first changed, on a vertical line to the left; a horizontal line is drawn under all. The quotient is evidently to be of degree 3 (the degree of the dividend minus that of the divisor), so that four spaces are counted off for the degrees 3, 2, 1, 0 and a vertical bar drawn. The first term of the quotient is written under the line, and its products by the terms of the divisor (except the first) on a left-to-right downward diagonal; the second column is summed and the process repeated until the products by the absolute term (that of degree zero) of the quotient have been written; then the columns are all summed and the result is read.

The process may be continued to develop the quotient in negative powers of x; then each column must be filled at the top before being summed.

The preceding exposition of synthetic division presupposes the arrangement of both dividend and divisor in descending powers of x, but the process is equally valid if they are arranged in ascending powers.

Example:

$$(x^4+4)\div(x^2-2x+2) \quad \text{or} \quad \left(2+\frac{x^4}{2}\right)\div\left(1-x+\frac{x^2}{2}\right).$$

1	1+0+0	+0+4		1	2+0+0	+0+½	
2	2+4	+4		1	2+2	+1	
−2	−2	−4−4		−½	−1	−1−½	
	1+2+2	+0+0			2+2+1	+0+0	

Result: (x^2+2x+2) or $(2+2x+x^2)$, the same.

68. Synthetic Substitution of a Number.—If $f(x)$ is a polynomial, its value for $x=a$ may be found by the process of synthetic division, since, if the given polynomial is divided by $(x-a)$, the remainder is $f(a)$.

For instance, given $f(x)\equiv x^3-7x^2+17x-2$, to find $f(2)$; we divide $f(x)$ by $(x-2)$ as follows:

2	1−7+17	− 2
	2−10	+14
	1−5+ 7;	+12

getting the remainder 12; hence, by the Remainder Theorem, $f(2)=12$.

Examples.

In each of the following examples use the synthetic method.

Remove the rational roots from the following, and solve the resulting reduced equation:

1. $3x^3-16x^2+12x+16=0$.
2. $12x^4-x^3-54x^2+4x+24=0$.
3. $3x^3-14x^2-25=0$.
4. $2x^4+x^3-13x^2-5x+12=0$.

Find the values of the following functions for the indicated values of x:

5. $2x^3-3x^2-36x+20$ for $x=-3, -2, -1, 0, 1, 2, 3$.
6. x^3+3x^2-9x+2 for $x=-4, -3, -2, -1, 0, 2$.

Perform the following divisions:

7. $2x^5-3x^4+x^3-5x+7$ by x^4-2x^3+2x-2.
8. $(x+1)^6$ by $(x-1)^3$.
9. $15x^6-6x^5-26x^4+2x^3-21x^2+28$ by $5x^3-2x^2+3x-4$.
10. $1+11x+11x^2+x^3$ by $(1-x)^5$ to six terms.
11. Expand $\sqrt{1+x}$ to six terms, and $\sqrt{1-x}$ to six terms, and obtain by division the expansion to six terms of $\sqrt{\frac{1+x}{1-x}}$; obtain the same result by dividing the expansion of $\sqrt{1-x^2}$ by $1-x$.

69. Synthetic Substitution of a Binomial.—Suppose it is required to substitute $(y+a)$ for x in a cubic function

$$ax^3+bx^2+cx+d;$$

the actual substitution is a long process, but we can imagine the powers of $(y+a)$ expanded and the terms of like degree combined, giving an expression

$$py^3+qy^2+ry+s.$$

Since $y=x-a$, if we should now substitute $(x-a)$ for y in this expression, again expand powers, and collect terms, we should have the original cubic function of x; then

$$p(x-a)^3+q(x-a)^2+r(x-a)+s\equiv ax^3+bx^2+cx+d.$$

Divide this identity by $(x-a)$; we know from the form of the left member that the remainder will be s; divide the quotient by $(x-a)$, and the remainder will be r; divide the second quotient by $(x-a)$, and the remainder will be q; it is unnecessary to go further, as it is evident that $p=a$.

Example: Given $f(x)\equiv 2x^3-15x^2+24x+6$, find $f(y+2)$.

$$\begin{array}{r|l} 2 & \begin{array}{rrrr} 2 & -15 & +24 & +6 \\ & +4 & -22 & +4 \\ \hline 2 & -11 & +2; & 10 \\ & +4 & -14 & \\ \hline 2 & -7; & -12 & \\ & +4 & & \\ \hline 2; & -3 & & \end{array} \end{array}$$

$$f(y+2)\equiv 2y^3-3y^2-12y+10.$$

Suppose the number r to be a root of $2x^3-15x^2+24x+6=0$; making $x=r$ is equivalent, since $x=y+2$, to making $y+2=r$, or $y=r-2$; hence $r-2$ will be a root of the new equation $2y^3-3y^2-12y+10=0$. The method and results are evidently general, so we have:

70. To Diminish the Roots of an Equation.—*Given $f(x)$, a polynomial, to find $f(y+a)$; or given an algebraic equation $f(x)=0$, to write a new equation in terms of y of which each root shall be less by a than the corresponding root of $f(x)=0$; divide $f(x)$ by $(x-a)$, taking the remainder as the new constant term; divide the quotient by $(x-a)$, taking the remainder as the coefficient of y, and so proceed, getting all the desired coefficients successively.*

71. Removing the Term of Next to the Highest Degree.—It is evident that in the process just described, if the roots of $ax^3+bx^2+cx+d=0$ are lessened by the amount $-\frac{b}{3a}$, the amount $\frac{b}{3}$ will be subtracted three times from b, making the coefficient of the y^2 term in the new equation zero. So for any equation, beginning $ax^n+bx^{n-1}+\ldots.$, if the values of the roots are lessened by $-\frac{b}{na}$, the quantity $\frac{b}{n}$ will be subtracted n times from b, leaving zero.

Example: Remove the squared term from

$$2x^3-12x^2+7x-3=0.$$

Evidently using $+2$ three times as a synthetic divisor will remove the term:

2	2	−12	+ 7	− 3
		4	−16	−18
	2	− 8	− 9 ;	−21
		4	− 8	
	2	− 4 ;	−17	
		4		
	2 ;	0		

The new equation is $2y^3-17y-21=0$, having roots less by 2 than the corresponding roots of the original equation.

Examples.

Diminish the roots of the following equations by the values indicated:

1. $3x^3-4x^2+7x-7=0$ by 1. *Ans.* $3y^3+5y^2+8y-1=0$.
2. $x^4-12x^3+47x^2-66x+30=0$ by 3; and solve.
Ans. Roots, 5, 1, $3\pm\sqrt{3}$.
3. $4x^4-8x^3+3x^2+x-3=0$ by $\frac{1}{2}$; and solve.
Ans. Roots, $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$, $\frac{1}{2}\pm\frac{1}{2}\sqrt{-2}$.

Increase the roots of the following equations by the values indicated:

4. $7x^3-11x^2+15x-6=0$ by 1.
Ans. $7y^3-32y^2+58y-39=0$.
5. $5x^3+10x^2+5x-1=0$ by 2.

Remove the term of the next to the highest degree from each of the following equations:

6. $7x^3+63x^2+34x-276=0$; and solve.
Ans. -3, $-3\pm\frac{1}{7}\sqrt{1085}$.
7. $x^3-5x^2+6x-1=0$.
8. $ax^2+bx+c=0$; and solve.

72. Cubic Equations.—We have already seen that the solution of any cubic equation may be made to depend upon the solution of an equation of the form $x^3+px+q=0$ (Art. 71). Scipione del Ferro, of Bologna (born 1465, died 1526), discovered the solution of this simpler form, which was afterwards lost. February 12, 1535, Nicolo Fontana (nicknamed Tartaglia, "the Stammerer") discovered and published the solution that follows, and by 1541 had learned to apply it to the solution of any cubic equation. Under a pledge of secrecy, Girolamo Cardan, of Pavia, secured from Tartaglia the general solution and straightway published it. The solution that follows is commonly called Cardan's. Cardan also published the solution of the general equation of the fourth degree, or biquadratic equation, discovered by his pupil Ludovico Ferrari, of Bologna (1522-1562).

73. The Cube Roots of Unity.—The simplest cubic equation is $x^3=1$ or $x^3-1=0$, which, since it is factorable, is readily solved:

$$(x-1)(x^2+x+1)=0$$

has for its roots

$$1, \quad \frac{-1\pm\sqrt{-3}}{2},$$

or

$$1, \quad \frac{-1+i\sqrt{3}}{2}, \quad \frac{-1-i\sqrt{3}}{2}.$$

If we denote $\frac{-1+i\sqrt{3}}{2}$ by ω, we find, by actually squaring ω and taking its reciprocal, that

$$\frac{-1-i\sqrt{3}}{2}=\omega^2=\frac{1}{\omega}.$$

As any number n is $n\times 1$, the cube root of n is

$$\sqrt[3]{n}, \quad \omega\sqrt[3]{n}, \text{ or } \omega^2\sqrt[3]{n},$$

$\sqrt[3]{n}$ denoting any number which, when cubed, gives n.

74. The General Cubic, $x^3+px+q=0$.—To solve

$$x^3+px+q=0. \tag{1}$$

Let

$$x=z+v; \tag{2}$$

then

$$z^3+3z^2v+3zv^2+v^3+p(z+v)+q=0,$$

or

$$z^3+v^3+(3zv+p)(z+v)+q=0. \tag{3}$$

The condition $x=z+v$ does not fix the values of z and v; we can impose a second condition on the two quantities, and will choose it so as to simplify equation (3).

Make $3zv+p=0$ by letting

$$v=-\frac{p}{3z}; \tag{4}$$

then (3) becomes

$$z^3+v^3+q=0, \tag{5}$$

or

$$z^3-\frac{p^3}{27z^3}+q=0,$$

or

$$z^6+qz^3=\frac{p^3}{27}. \tag{6}$$

Solving (6) as a quadratic in z^3, we get

$$z^3=-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}; \tag{7}$$

and from (5),

$$v^3=-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}. \tag{8}$$

$x=z+v$, and whether we take the upper or the lower signs,

$$x=\left(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

There are three cube roots of z^3 and three of v^3; these can be combined in only three ways (Art. 73). However, as from (4) it is evident that the product of a pair is always the same, $-\frac{p}{3}$, if the cube roots of z^3 are z, ωz, and $\omega^2 z$, and if $\frac{p}{-3z}=v$, then

$$\frac{p}{-3\omega z}=\omega^2 v, \quad \frac{p}{-3\omega^2 z}=\omega v,$$

since $\omega^3=1$, and the three roots of $x^3+px+q=0$ are

$$z+v,\ \omega z+\omega^2 v, \text{ and } \omega^2 z+\omega v.$$

Evidently, if $\frac{q^2}{4}+\frac{p^3}{27}$ is negative, it is necessary to find the cube root of an imaginary in order to solve the equation. This can be done, approximately, by methods of trigonometry, but not by arithmetical processes. The solution is therefore not general in the same sense as that of the quadratic equation. It can be shown that it is precisely in this *irreducible* case (when $\frac{q^2}{4}+\frac{p^3}{27}<0$) that an equation $x^3+px+q=0$ with real coefficients has three real roots.

A perfectly general method for obtaining the real roots of any algebraic equation having real coefficients will be developed later under the name of *Horner's Method.*

75. As an illustration of *Cardan's Method,* we will solve the equation

$$x^3-15x+30=0.$$

Here

$$\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=\overline{15}^2+(-5)^3>0.$$

Let $x=z+v$;

$$z^3+v^3+(3zv-15)(z+v)+30=0.$$

Let $zv=5$;

$$z^3+v^3+30=0,$$

$$z^3+\frac{125}{z^3}+30=0,$$

$$z^6+30z^3+125=0.$$

$$z^3=-\frac{30\pm 20}{2}=-5 \text{ or } -25.$$

$$v^3=-30-z^3=-25 \text{ or } -5.$$

$x=-\sqrt[3]{5}-\sqrt[3]{25}$; reducing by logarithms,

$$\sqrt[3]{25}=2.9240, \quad \sqrt[3]{5}=1.7100;$$

$x=-1.7100-2.9240=-4.6340$, the real root.

The imaginary roots are

$$-\omega\sqrt[3]{5}-\omega^2\sqrt[3]{25} \text{ and } -\omega^2\sqrt[3]{5}-\omega\sqrt[3]{25},$$

which, simplified, are

$$\tfrac{1}{2}(\sqrt[3]{25}+\sqrt[3]{5}) \pm \tfrac{1}{2}(\sqrt[3]{25}-\sqrt[3]{5})\sqrt{3}\,i,$$

or

$$2.3170 \pm 1.0514\sqrt{-1}.$$

The sum of the three roots is clearly zero; the product is

$$-4.6340(\overline{2.3170}^2+\overline{1.0514}^2) = 30.000.$$

76. Trigonometric Solution of the Cubic with Three Real Roots.—In finding the three real roots of an equation

$$x^3+px+q=0,$$

$\left(\frac{q^2}{4}+\frac{p^3}{27}\right)<0$, we must abandon the general solution altogether, and make use of the identity proved in trigonometry:

$$\cos^3\theta-\tfrac{3}{4}\cos\theta-\tfrac{1}{4}\cos 3\theta \equiv 0.$$

Now in $x^3+px+q=0$ we can assume $x=n\cos\theta$, getting

$$\cos^3\theta+\frac{p}{n^2}\cos\theta+\frac{q}{n^3}=0,$$

which, according to the identity, will hold if

$$\frac{p}{n^2}=-\tfrac{3}{4} \quad \text{or} \quad n=\sqrt{\frac{-4p}{3}},$$

and

$$\frac{q}{n^3}=-\frac{\cos 3\theta}{4} \quad \text{or} \quad \cos 3\theta=-\frac{4q}{n^3}.$$

We compute $\cos 3\theta$, and thence find 3θ and θ.

It is shown in trigonometry that $360^\circ \pm 3\theta$ has the same cosine as 3θ, so the three real roots are

$$x_1=n\cos\theta, \quad x_2=n\cos(120^\circ+\theta), \quad x_3=n\cos(120^\circ-\theta).$$

As a check, we have

$$x_1+x_2+x_3=0, \quad x_1x_2x_3=-q.$$

77. *Example:* Solve $x^3-6.75x+1.75=0$.

In this example, it is to be noted that the cosine of an obtuse angle is negative; i. e.,

$$\cos(180°-\phi)\equiv-\cos\phi.$$

A convenient form for the solution is:

$-4q=-7$	$\log\ 0.84510_n$
$n=\sqrt{-\frac{4}{3}p}=3. \quad n^3=27$	$\operatorname{colog}\ 8.56864-10$
$3\theta=105°\ 01'\ 34.5''$	$\log\cos\ 9.41374-10_n$

$\theta=$	$35°00'31.5''$	$\log\cos 9.91332-10$	
$120°+\theta=$	$155°00'31.5''$	$\log\cos 9.95730-10_n$	
$120°-\theta=$	$84°59'28.5''$		$\log\cos 8.94106-10$
$-\frac{4p}{3}=n^2$		$\frac{1}{2}\log 0.47712$	$\frac{1}{2}\log 0.47712$	$\frac{1}{2}\log 0.47712$
$x_1=$	2.4572	$\log 0.39044$		
$x_2=$	-2.7191	$\log 0.43442_n$	$\log 0.43442_n$	
$x_3=$	0.26192	$\log 9.41818$		$\log 9.41818$
$x_1x_2x_3=-q=$	-1.75	$\log 0.24304_n$		
$x_1+x_2+x_3=$	0.0000			

We have finally two methods of solving $x^3+px+q=0$:

Cardan's Algebraic Solution when $\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3>0$.

Trigonometric Solution when $\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3<0$.

Clearly the roots are $2\sqrt[3]{-\frac{q}{2}}$, $\sqrt[3]{\frac{q}{2}}$, $\sqrt[3]{\frac{q}{2}}$ when

$$\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=0. \quad \text{(Note that } \omega+\omega^2=-1.\text{)}$$

Examples.

1. Find the real root of $x^3-5x+5=0$. *Ans.* -2.6274.
2. Find the real root of $x^3+2x-3.5=0$. *Ans.* 1.0945.
3. Find all the roots of $x^3-6x+5.2=0$.
   Ans. -2.8027, 1.0719, 1.7308.
4. Find all the roots of $x^3-6x-5.2=0$.
   Ans. -1.7308, -1.0719, 2.8027.
5. Find all the roots of $x^3-4.5x-2.55=0$.
   Ans. -1.7426, -0.6195, 2.3621.
6. Solve $x^3-x^2+x+1=0$ and check.
7. Solve $2x^3+3x^2-12x-10=0$ and check.
8. Solve $x^3-3x^2-9x+15=0$ and check.
9. Solve $x^3-12x+16=0$ by Cardan's Method.
10. Solve Example 9 by method of Art. 76.

CHAPTER V.

Graphical Representation.

78. Representation of Real Numbers.—For purposes of comparison real numbers are represented on a scale as follows:

On a straight line mark a zero-point O, and choosing some convenient length to correspond to a unit of numerical magnitude, represent every *positive* number by a point to the *right* of 0, and every *negative* number by a point to the *left*, making the distance from 0 to the point correspond to the numerical magnitude of the number.

This process is called *plotting* the number.

Fig. 2 shows a scale on which a few numbers have been plotted. Square roots of positive integers are readily plotted without computation, and their values roughly estimated from their positions on the scale.

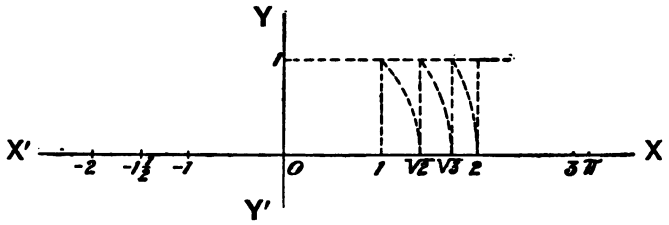


Fig. 2.

79. Plotting the Graph of a Function.—We may show graphically by means of a curve the values of a function corresponding to given values of its independent variable, and also the way in which a function changes as its variable is changed.

Draw two straight lines, $X'X$ and $Y'Y$, at right angles, inter-

secting at O. Taking O as the origin of measurement, or zero-point, lay off, on any convenient scale, the values of x along $X'X$, positive to the right, negative to the left, as described in the preceding article. The line $X'X$ we shall call the *x-axis*.

At each point of the x-axis, erect a perpendicular line of length equal by scale to the magnitude of $f(x)$ corresponding to the value of x at that point, *above* $X'X$ if the function is positive, *below* if it is negative; draw a smooth curve through the ends of the perpendiculars. This curve is called the *graph* of $f(x)$.

The perpendicular distance from any given point of the graph to the x-axis is called the *ordinate* of the point, and the value of x for the given point is called the *abscissa*.

The value of $f(x)$ for any given value of x is the length by scale of the *ordinate* of the curve corresponding to the given value of x. Also, the value of x that causes $f(x)$ to have any given value is the length by scale of the *abscissa* of that point of the graph whose ordinate is the given value of $f(x)$.

80. Graph of a Linear Function.—If $f(x)$ is of the first degree, its graph is a straight line, and therefore such a function is called *linear*.

For example, given

$$f(x) = \frac{x}{2} + 2,$$

the values of $f(x)$, for values of x from -4 to $+3$, are:

x :	-4,	-3,	-2,	-1,	0,	$+1$,	$+2$,	$+3$
$f(x)$:	0,	$+\frac{1}{2}$,	$+1$,	$+\frac{3}{2}$,	$+2$,	$+\frac{5}{2}$,	$+3$,	$+\frac{7}{2}$

The points corresponding to these values are plotted in Fig. 3, and the graph drawn through them.

If a straight line is drawn from each point of the graph parallel to the x-axis, as in the figure, it is obvious, from the

successive values of $f(x)$, that the perpendicular in the series of right triangles thus formed is $\frac{1}{2}$ the base. Hence all the right triangles are equal, and the acute angles adjacent to their bases are equal. From the principles of plane geometry this can only be true if the successive portions of the graph lie in the same straight line, AB.

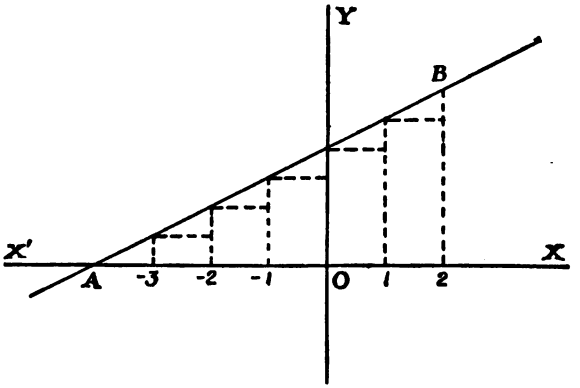


FIG. 3.

A further deduction may be made from this property, that *the increase in $f(x)$ bears a constant ratio to the increase in x.*

In this particular case the ratio is $\frac{1}{2}$, the coefficient of x in $f(x)$, and in any other linear function $(mx+b)$, the ratio would be m. This ratio measures the *tangent* of the angle which the straight line makes with the x-axis.

Examples.

Draw the graphs of the following functions:

1. $3x$, $2x$, x, $\frac{1}{2}x$, $\frac{1}{3}x$.
2. $-2x$, $-\frac{1}{2}x$, $-2x+3$, $-\frac{1}{2}x+2$.
3. Using the same set of axes, draw the graphs of $5x-2$ and $3x+4$. How do the graphs indicate the value of x for which the two functions are equal?

81. Graph of x^2.—We shall find that the graph of any other function of x than the linear function is a curve. Consider the graph of x^2. When

$$x = -3, -2, -\tfrac{3}{2}, -1, -\tfrac{3}{4}, -\tfrac{1}{2}, -\tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{3}{2}, 2, 3,$$

$$f(x) \equiv x^2 = 9, 4, \tfrac{9}{4}, 1, \tfrac{9}{16}, \tfrac{1}{4}, \tfrac{1}{16}, 0, \tfrac{1}{16}, \tfrac{1}{4}, \tfrac{9}{16}, 1, \tfrac{9}{4}, 4, 9.$$

Plotting these values by the process described, and sketching in a smooth curve through the successive points, we obtain the graph given in Fig. 4.

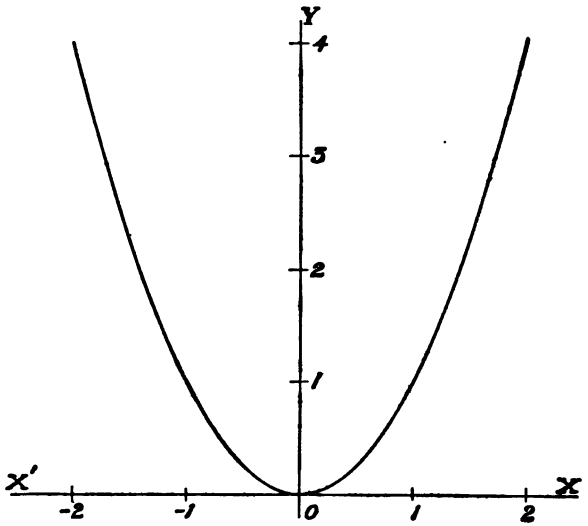


FIG. 4.

The abscissa and ordinate of any point on this graph represent a number and its square, so that we can determine the square of a number a by measuring the ordinate of the point of the graph having the abscissa a, or we can determine the square root of a number b by measuring the abscissa of the point of the graph having the ordinate b.

Any graph shows the real values of the function corresponding to real values of the variable; this graph shows that every real number has a positive square, that a negative number has no real square root, and that a positive number has two real square roots, numerically equal, but opposite in sign.

The graph also shows that if x starts with a negative value and increases, x^2 decreases, rapidly at first, then more and more slowly until it becomes zero when $x=0$; from here x^2 increases as x increases, slowly at first, then more and more rapidly.

82. The Graphs of (x^2+a) and $(x+a)^2$.—By properly shifting the axes, we can make the same curve that we drew as the graph of x^2 serve as the graph of (x^2+a) or of $(x+a)^2$, a being any given number. For instance, suppose we should construct the graph of (x^2+1) independently. For any value of x, the point representing (x^2+1) would be one unit higher than the point representing x^2; thus the graph of (x^2+1) would be the graph of x^2 raised one unit. We can, however, get the same effect more easily by leaving the graph where it is and *lowering* the x-axis one unit. In the same way, the graph of (x^2-2) can be obtained from the graph of x^2 by raising the x-axis two units.

Again, suppose we should construct the graph of $(x+1)^2$; in the table of values of x and $(x+1)^2$, we should have the same set of values for $(x+1)^2$ that we had for x^2, except that the corresponding value of x would be one less. This means that the graph of $(x+1)^2$ is the graph of x^2 shifted one unit to the *left,* or is the graph of x^2 with the y-axis shifted one unit to the *right.* In the same way, the graph of $(x-2)^2$ can be obtained from the graph of x^2 by shifting the y-axis two units to the *left.*

These considerations are perfectly general, and may be applied to any function:

The graph of $f(x)+a$ can be obtained from the graph of $f(x)$ by shifting the x-axis a units downward; and

The graph of $f(x+a)$ can be obtained from the graph of $f(x)$ by shifting the y-axis a units to the right.

If a is negative, the shift is in the opposite direction.

Examples.

Draw the graphs of the following functions:

1. x^2-3. 2. $2x^2+1$. 3. $-\frac{1}{2}x^2$. 4. x^3. 5. $-2x^3$.
6. x^2+x^3.
7. x^2 and $2x+3$ on the same axes. How do the graphs show the values of x for which these two functions are equal?
8. x^2 and $2x-1$ on the same axes. How do the graphs show the values of x for which $x^2-2x+1=0$?

83. Slope of the Graph.—The steepness or slope of a graph at any point, which is the same as the slope of the tangent at that point, measures the rapidity with which the function is increasing with respect to its variable. The slope of a straight line is the tangent of the angle made by the line with the axis of x, and is considered positive if the acute angle made above the x-axis by these lines opens to the right, negative if it opens to the left. A decrease is thus regarded as a negative increase. We have already seen the use of the slope in the case of the linear function, $mx+b$, for which the slope m of the graph showed that when x was increased, $mx+b$ was increased m times as much.

84. Slope of the Graph of x^2.—Let P be any point of the graph of x^2, having the abscissa x and the ordinate x^2. Let PT represent the tangent to the graph at P, and draw PM parallel to OX, of any convenient length; erect a perpendicular to PM at M, meeting PT at T.

The ratio $\frac{MT}{PM}$ is the desired slope of the graph (or of its tangent PT) at the point P; in order to find its value, erect a perpendicular to PM at any point N, extending it to meet the graph at Q. Draw the secant PQ and let it meet MT at S.

The ratio $\frac{NQ}{PN}$ is by similar triangles equal to the ratio $\frac{MS}{PM}$. If we suppose the point N to move along MP, approaching P, Q will move along the graph, approaching P, and the lengths NP and NQ will each approach zero as a limit; the ratio $\frac{NQ}{PN}$ will thus become indeterminate, of the form $\frac{0}{0}$, but as for every position it is equal to $\frac{MS}{PM}$, it must approach the same limit as $\frac{MS}{PM}$, or $\frac{MT}{PM}$. Then if we can find the limit approached by $\frac{NQ}{PN}$, we shall have the desired slope, $\frac{MT}{PM}$.

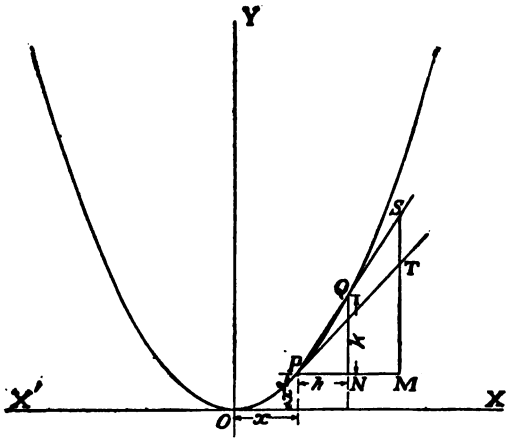


FIG. 5.

For convenience, call $PN=h$, $NQ=k$; the abscissa and ordinate of P are x and x^2 respectively; those of Q are $(x+h)$ and (x^2+k). Since $(x^2+k)=(x+h)^2$,

$$k=x^2+2hx+h^2-x^2=2hx+h^2.$$

The variable ratio $\frac{NQ}{PN} = \frac{k}{h} = \frac{2hx+h^2}{h}$; we want the limit, $\frac{MT}{PM}$, or

$$\left[\frac{k}{h}\right]_{h=0} = \frac{2hx+h^2}{h}\Big]_{h=0} = 2x+h]_{h=0} = 2x.$$

Hence $\frac{MT}{PM}$, the desired slope, is $2x$.

85. From this general value of the slope we can derive the properties of the graph and the function that we observed in the discussion of Fig. 4; when x is negative and numerically large, the slope $2x$ indicates that the graph slopes steeply downward to the right, or that x^2 decreases rapidly as x increases, and so on. Note particularly that the slope changes sign as x passes through the value zero, and becomes zero when $x=0$, indicating that the graph has the direction of the x-axis, and that the function has a minimum value. Besides this, the value $2x$ gives the slope of the graph for any value of x and shows the rate at which the function is increasing relatively to its argument x. For instance, when $x=\frac{1}{2}$, the slope is 1; that is, the tangent to the graph makes an angle of 45° with the x-axis, and the function x^2 is here increasing just as fast as x.

86. The Derivative.—The slope in terms of x of the graph of any function of x is called the *derivative of the function with respect to x.*

When the derivative is negative, the function decreases as x increases; when the derivative is positive, the function increases as x increases; when the derivative is zero, the graph of the function is parallel to the axis of x and the function has a minimum or a maximum value.

When a function of x is denoted by $f(x)$, its derivative with respect to x is denoted by $f'(x)$.

The process used to find the derivative of x^2 will give the derivative of any function of x. A careful graph of the function may be drawn in each case, but Fig. 5 will serve perfectly well as a diagram for all cases. The first part of the discussion is general; for any function, $f(x)$, the derivative $f'(x)$, or the slope of the tangent PT, is

$$f'(x) = \frac{MT}{PM} = \left[\frac{NQ}{PN}\right]_{PN=0} = \left[\frac{k}{h}\right]_{h=0}.$$

In finding the derivative of x^3, we have x and x^3 for the abscissa and ordinate of P. For those of Q, we have $(x+h)$ and $(x^3+k)=(x+h)^3$; here

$$k = 3hx^2 + 3h^2x + h^3;$$

$$\left[\frac{k}{h}\right]_{h=0} = \left[\frac{3hx^2+3h^2x+h^3}{h}\right]_{h=0} = 3x^2+3hx+h^2]_{h=0} = 3x^2.$$

Thus the derivative of x^3 is $3x^2$; the derivative of x^4 is found in the same way to be $4x^3$; and so on.

If we expand $(x+h)^n$ by the binomial theorem, we shall find in the same way that *the derivative of* x^n *is* nx^{n-1}.

Examples.

1. Find the derivatives of x^2+2, of $2x^2$.
2. Find the derivatives of x^3+4, of $5x^3$.
3. Find the derivative of x^4. What are the derivatives of ax^4 and of x^4+a, if a is constant? What is the slope of the graph of x^4 when $x=\frac{1}{2}$?
4. Find the derivative of x^2+x^3.

87. Derivative of a Polynomial.—When we find the derivative of x^n+a, where a is any constant, the value a drops out, and

when we find the derivative of ax^n, the factor a persists throughout the process, so that:

The derivative of x^n+a is the same as that of x^n,

or

The derivative of a constant is zero;

and

The derivative of ax^n is nax^{n-1}.

When we find the derivative of the sum of two or more functions, the process gives the sum of the derivatives of the several functions.

We have consequently the following rule for finding the derivative of any polynomial:

Multiply each term by the exponent of x in that term, and reduce the exponent of x one unit. Drop any constant term.

The constant term may be regarded as the coefficient of x^0, and the last part of the rule omitted.

88. Graph of the Polynomial of Degree n.—Consider the graph of the polynomial

$$f(x) \equiv \tfrac{1}{10}(4x^3-6x^2-45x+12).$$

The derivative is

$$f'(x) \equiv \tfrac{1}{10}(12x^2-12x-45).$$

The derivative is zero when $x=\frac{5}{2}$ or $-\frac{3}{2}$; the corresponding points of the graph are important.

The values of $f(x)$ are found by synthetic division; those that we shall need are the integral values from -4 to $+5$ and the values when $f'(x)=0$; these are:

$x=$	-4,	-3,	-2,	$-\frac{3}{2}$,	-1,	0,	1,	2,	$\frac{5}{2}$,	3,	4,	5;
$f(x)=$	-16.0,	-1.5,	4.6,	5.25,	4.7,	1.2,	-3.5,	-7.0,	-7.55,	-6.9,	-0.8,	13.7;
$f'(x)=$	pos.,	pos.,	pos.,	0,	neg.,	neg.,	neg.,	neg.,	0,	pos.,	pos.,	pos.

Plotting these values of $f(x)$, and sketching a smooth curve through the successive points, we obtain the graph given in Fig. 6.

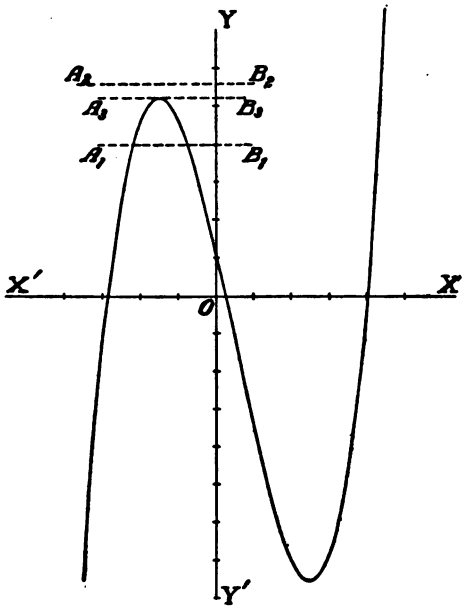


FIG. 6.

89. Use of the Graph.—We study the relations between the value of $f(x)$ and its independent variable, as x increases gradually from -4 to $+5$, by following the extremity of the ordinate which represents the value of $f(x)$, as it slides along the graph.

When $x=-4$, $f(x)$ starts with the value -16; and as x increases, $f(x)$ increases, passing through the value 0 when x is about -2.8, until, when $x=-1.5$, it reaches its highest value for the immediate vicinity, or reaches a *maximum*. It then begins to decrease, passing again through the value zero when $x=$about 0.2, until when $x=2.5$ it reaches its lowest value for the immediate vicinity, or reaches a *minimum*. It then begins

to increase, passing once more through the value zero when x = about 4.1.

90. Relation of Graph of $f(x)$ to Solution of $f(x)=0$.—The graph of a function $f(x)$ is chiefly used, in algebra, as an aid to the solution of the equation $f(x)=0$. From the graph of

$$f(x) \equiv \tfrac{1}{10}(4x^3-6x^2-45x+12)$$

just considered, it is evident that the values of x for which $f(x)=0$ are roughly -2.8, $+0.2$, $+4.1$; that is, these are the approximate roots of $f(x)=0$.

If the absolute term of $f(x)$ is decreased by a units, the graph of the resulting function,

$$f_1(x) \equiv \tfrac{1}{10}(4x^3-6x^2-45x+12-10a),$$

may be got by *moving the axis upward a units.* This process will change the roots, but can not change their number, since $f(x)$ and $f_1(x)$ are of the same degree.

If $a=4$, we clearly have two negative roots in the vicinity of the maximum point, as the new x-axis A_1B_1 cuts across the curve just below that point; if $a=5.5$, the new x-axis A_2B_2 passes above the maximum point, and only one real root exists, the left-hand roots having disappeared. If $a=5.25$, the axis will be tangent to the curve at the maximum point, and there the points indicating the two roots will merge into *one.* The graph indicates, in these three cases, *two distinct, two imaginary,* and *two equal roots,* respectively, by its appearance in the vicinity of this maximum point.

If, however, the y-axis (i. e., the *origin*) be shifted to the right or left, the magnitude of the roots will be diminished or increased, but the character of the roots, whether real or imaginary, remains the same. If the origin be shifted one unit to the right, the values of x for the points where the curve crosses the x-axis are changed by -1.0, from -2.8 to -3.8, from $+0.2$ to -0.8, from $+4.1$ to $+3.1$. With the y-axis in this position, the curve is the graph of $f(x+1) \equiv \tfrac{1}{10}(4x^3+6x^2-45x-35)$.

91. Multiple Roots.—*When the axis of abscissas is tangent to the graph,* i. e., *when* $f(x)$ *and* $f'(x)$ *are zero for the same value of* x, *this value is a double root of* $f(x)=0$.

In fact, if $f(r)=f'(r)=0$, r is *at least* a double root of $f(x)=0$; it may occur more than twice as a root.

It can be shown that if the common root r occurs k times as a root of $f'(x)=0$, it is a root $(k+1)$ times of $f(x)=0$.

We may thus determine the equal roots of an equation by finding all the factors common to the functions $f(x)$ and $f'(x)$, which is most conveniently done by the process of finding the greatest common divisor.

For example, find the equal roots of the equation

$$f(x)=x^4-6x^3+12x^2-10x+3.$$

The derivative of $f(x)$ is

$$f'(x)=4x^3-18x^2+24x-10.$$

Their G. C. D., by detached coefficients, is found below:

$$\begin{array}{rrrrr|rrrr}
1- & 6+ & 12- & 10+ & 3 & 2- & 9+ & 12- & 5 \\
2- & 12+ & 24- & 20+ & 6 & 2- & 4+ & 2 & \\
\cline{6-8}
2- & 9+ & 12- & 5 & & & 5- & 10+ & 5 \\
\cline{6-8}
\multicolumn{2}{r}{3-} & 12+ & 15- & 6 & & 5- & 10+ & 5 \\
\cline{6-8}
\multicolumn{2}{r}{6-} & 24+ & 30- & 12 & & & & \\
\multicolumn{2}{r}{6-} & 27+ & 36- & 15 & & & & \\
\cline{6-8}
\multicolumn{3}{r}{3-} & 6+ & 3 & & & & \\
\multicolumn{3}{r}{1-} & 2+ & 1 & & & &
\end{array}$$

G. C. D.

Hence,

$$f'(x)\equiv(x-1)^2(4x-10),$$
$$f(x)\equiv(x-1)^3(x-3),$$

and the equal roots of $f(x)=0$ are 1, 1, 1.

Examples.

1. $f(x)=x^3-5x^2+3x+9$; determine the maximum and minimum points, and equal roots, if any.

2. Construct the graph for $x^3-2x^2-5x+5=0$, first finding the points of maximum and minimum values.

3. Show that the equation $x^4-10x^3+36x^2-54x+27=0$ has equal roots, and find them.

4. Find the minimum point of the quadratic polynomial (x^2+2x-4) and draw the graph.

5. Construct the graphs for the following, and estimate the roots of each equation to the nearest tenth (draw only one curve):

$3x^3+6x^2-5x+7=0$, $\quad 3x^3+6x^2-5x+1=0$,
$3x^3+6x^2-5x-6=0$, $\quad 3x^3+6x^2-5x-11=0$,
$3x^3+6x^2-5x-12=0$, $\quad 3x^3+6x^2-5x-17=0$.

6. What constant term would give equal roots to the equation $3x^3+6x^2-5x+a=0$?

7. If $f(x)\equiv ax^2+bx+c$, find the minimum value of $f(x)$, a being positive, and the conditions that this minimum shall be negative, zero, or positive.

8. Construct the graph for $x^3+2x^2+3x-4=0$ and locate the roots approximately.

9. Find the maximum and minimum values of x^3+px+q. What is the condition that two of the roots of $x^3+px+q=0$ be equal?

10. How may the two graphs of x^3 and $(-px-q)$ be used to show the roots of $x^3=-px-q$? Apply to Example 8.

CHAPTER VI.

Horner's Solution of Equations.

92. Horner's Method.—Suppose that we wish to find a root of the equation

$$f(x) \equiv 2x^3 - 15x^2 + 24x + 6 = 0.$$

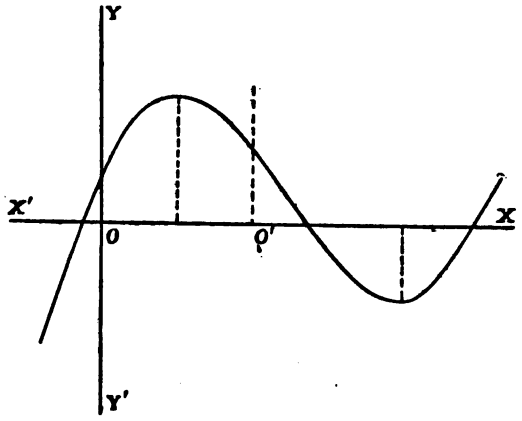


Fig. 7.

Any constant multiple of $f(x)$ is zero whenever $f(x) = 0$, and only then; so we first draw the graph of $k \cdot f(x)$, giving k any convenient value. In the figure, $k = \frac{1}{10}$. From the graph, it appears that there is a root of $f(x) = 0$ between -1 and 0, one between 2 and 3, and one between 4 and 5. We will compute the root between 2 and 3.

We first deduce the equation with roots each 2 less than those of $f(x) = 0$; its graph will be the curve already drawn if the

axis of ordinates is moved 2 units to the right from O to O'; the new equation has a root between 0 and 1, corresponding to the root of $f(x)$ between 2 and 3. If this root is s, we have, substituting it in the new equation,

$$f_1(s) \equiv 2s^3 - 3s^2 - 12s + 10 = 0.$$

2)	2	−15	+24 + 6
		4	−22 + 4
		−11	+ 2;+10
		4	−14
		− 7	;−12
		4	
	2	;− 3	

Now as s is less than 1, its square and cube are much smaller, and we shall come pretty near to finding its true value by neglecting the first terms and solving $-12s+10=0$, whence $s=.8$, about. Trying .8, we find that $f_1(.8)$ is *negative*, showing, by the graph, that .8 is greater than the root of $f_1(s)$, because the corresponding point of the graph is below the x-axis. It is our object to take as large a value as we can, and still keep to the left of the root. $f_1(.7)$ is positive, hence the root of $f_1(s)=0$ is between .7 and .8, and the corresponding root of $f(x)=0$ is between 2.7 and 2.8.

We next derive the equation having a root .7 less than the root of $f_1(s)=0$, or 2.7 less than the root of $f(x)=0$, and so proceed progressively, getting finally as close an approximation to the actual root as we desire. We shall no longer need to try two values at each step, as with the decrease of the value substituted, the neglect of the terms of higher degree will cease to affect the single figure that we desire to obtain.

We shall use the principle of contraction, retaining only as many decimal places in the last column as we desire in the root, and using only as many in the earlier columns as are needed to determine those in the last. The coefficients of the successive reduced equations may be marked, in order to keep track of the work.

The work for seven figures (six decimal places) follows:

			2.761921+
1	−7.5	+12	+3
	+2	−11	2
	−5.5	+ 1	(1) +5.000
	+2	− 7	−4.592
	−3.5	(1) − 6.00	(2) +0.408000
	+2	− .56	− .395424
(1)	−1.5	− 6.56	(3) +0.012576
	+ .7	− 7	− 6546
	−0.8	(2) − 6.6300	(4) +0.006030
	+ .7	+ .0396	− 5890
	−0.1	− 6.5904	+0.000140
	+ .7	+ 432	− 131
(2)	+0.66	(3) − 6.5472	+0.000009
	6	+ 8	− 7
	0.72	− 6.5464	+0.000002
	6	+ 08	
(3)	0.781	(4) − 6.5456	
	.782	+ 7	
(4)	.783	− 6.545	

One root of $f(x) = 0$ is therefore 2.761921, correct to six places of decimals.

93. In getting the root between 4 and 5, the absolute term in the first reduced equation will be negative, and each succeeding reduced equation must also have its absolute term negative. The work follows:

					4.9571947
1	−7.5		+12		+3
	+4		−14		−8
	−3.5		− 2	(1)	−5.000
	+4		+ 2		+4.374
	+0.5	(1)	0.00	(2)	−0.626000
	4		4.86		+0.544625
(1)	4.5		4.86	(3)	−0.081375
	.9		5.67		+ 79163
	5.4	(2)	10.5300	(4)	−0.002212
	.9		.3625		+ 1136
	6.3		10.8925		−0.001076
	.9		.3650		+ 1023
(2)	7.25	(3)	11.2575		−0.000053
	5		515		+ 45
	7.30		11.3090		−0.000008
	5		515		
(3)	7.357	(4)	11.3605		
	7.364		7		
(4)	7.371		11.3612		

If the attempt is made to try too large a figure at any stage, the error will appear through the last coefficient's coming out of the wrong sign, except in very bothersome equations with roots so near together that it is easy to skip from the vicinity of one to the vicinity of another. When the trial division for a new figure of the root gives more than 9 in the decimal place for that figure, the indication is that too small a figure was used in the preceding step. A doubtful case can be definitely settled by trying a figure one unit smaller than the proposed figure and another one unit larger, as was done at the start of the preceding example.

The method described above is known as *Horner's Method* for the approximation of numerical roots of algebraic equations.

94. Negative Roots.—If we wish to find one of the negative roots of an equation, we may carry on the process of successive synthetic divisions as with a positive root, taking care to guard against error in the signs of operations with negative quantities.

It is usually more convenient to change the equation into one whose roots are each the negative of the corresponding roots of the given equation. This is done by *changing the sign of odd or even powers of the variable.* (Art. 64a.)

The roots of

$$2y^3+15y^2+24y-6=0$$

are equal to those of the equation

$$2x^3-15x^2+24x+6=0$$

with the signs changed; for every value of x in the given equation, we shall have a corresponding value $y=-x$ in the transformed equation.

95. Extraction of Roots of Numbers.—To find the nth root of any number a, we may solve by Horner's method the equation $x^n - a = 0$.

The process, for example, of finding the cube root of 10, is to solve the equation $x^3 - 10 = 0$.

			2.1544347
0	0		−10
2	4		+ 8
4	8	(1)	− 2.000
(1) 6	(1) 12.00		+ 1.261
6.1	.61	(2)	− 0.739000
6.2	12.61		+ .677375
(2) 6.3	.62	(3)	− .061625
6.35	(2) 13.2300		+ 55573
6.40	.3175	(4)	− .006052
(3) 6.45	13.5475		+ 5569
6.454	.3200	(5)	− .000483
6.458	(3) 13.8675		+ 418
(4) 6.4624	258		− .000065
	13.8933		+ 56
	258		− .000009
	(4) 13.9191		
	26		
	13.9217		
	26		
	13.924		

CHAPTER VII.

Loci of Equations.

96. Implicit Functions.—If an equation in two variables, x and y, is solved so as to give an expression for y in terms of x, y is thereby expressed as a function of x. y is said to be given *implicitly* as a function of x by the original equation, and after the solution, y is said to be expressed *explicitly* as a function of x.

For instance, the equation

$$2x+3y-7=0$$

gives y as an *implicit* function of x; after solution,

$$y=\frac{7-2x}{3}$$

gives y as an *explicit* function of x.

Again,

$$x^2+y^2=25$$

expresses y as an implicit function of x;

$$y=\pm\sqrt{25-x^2}$$

expresses y as an explicit function of x.

97. Graph of an Implicit Function; Locus of an Equation; Equation of a Curve.—Suppose we have the equation

$$2x+3y-7=0,$$

and solve it, getting $y=\frac{7-2x}{3}$, and construct the graph of $\frac{7-2x}{3}$, the straight line AB, indefinite in extent, shown in Fig. 8 (Arts. 80 and 83).

From the construction of the graph, if x is the abscissa and y the ordinate of any point P of AB, then $y=\frac{7-2x}{3}$; but if x and y are abscissa and ordinate of any point Q, not on AB, y is either greater or less than $\frac{7-2x}{3}$. Then it is true of every point on the line AB that its abscissa x and its ordinate y satisfy the relation $y=\frac{7-2x}{3}$ or the relation $2x+3y-7=0$; moreover, this is true of no point that is not on AB.

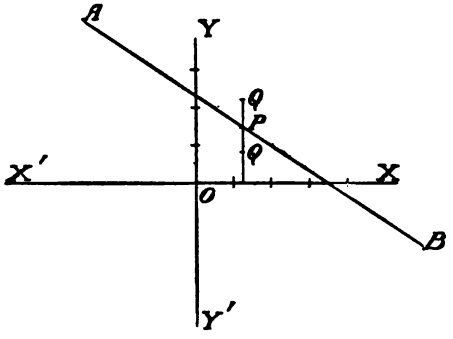


FIG. 8.

AB is consequently called the *locus of the equation*

$$2x+3y-7=0;$$

this is a brief way of calling it the locus of points whose coordinates satisfy the relation $2x+3y-7=0$.

$2x+3y-7=0$ is called the *equation of the straight line* AB.

A point with the abscissa x and the ordinate y is generally spoken of as the point having the coordinates x and y, or as the point (x, y); the abscissa is always mentioned first.

98. Relations between a Curve and its Equation.—The considerations of the preceding article are perfectly general, and may be applied to any case as follows:

If $f(x, y)=0$ when solved gives $y=F(x)$, and the graph of $F(x)$ is a certain curve:

The curve is the locus of $f(x, y)=0$;

$f(x, y)=0$ *is the equation of the curve;*

The coordinates of any point of the curve satisfy $f(x, y)=0$, *and the coordinates of any other point do not;* in other words,

Any point whose coordinates satisfy $f(x, y)=0$ *lies on the curve, and any point whose coordinates do not satisfy* $f(x, y)=0$ *does not lie on the curve.*

The relation between a curve and its equation may also be interpreted as follows: A point (x, y) will move all over the plane as x and y are varied; but if the restriction be put on x and y that $f(x, y)=0$, the point can move only along the curve. And again, as a point (x, y) moves about in the plane, the value of $f(x, y)$ will change, and will be positive when the point is on one side of the curve, negative when it is on the other side, and zero when it is on the curve.

It is common practice, instead of saying "the curve whose equation is $f(x, y)=0$," to say "the curve $f(x, y)=0$."

99. The Linear Equation and the Straight Line.—The most general equation of the first degree in two variables is

$$Ax+By+C=0,$$

where A, B, C are constants. On solution, it becomes

$$y=-\frac{A}{B}x-\frac{C}{B},$$

which is in the form

$$y=mx+b,$$

and so represents a straight line having the direction-ratio or slope $m=-\frac{A}{B}$, and cutting the y-axis at the point $\left(0, -\frac{C}{B}\right)$.

There are three special cases:

When $C=0$; then $Ax+By=0$ is satisfied by the coordinates of the origin, which is therefore a point of the line.

When $A=0$; then $y=-\frac{C}{B}$; and any point of the line must be at the distance $-\frac{C}{B}$ from the x-axis, or the line is a parallel to the x-axis and distant $-\frac{C}{B}$ units from it. Here $m=-\frac{A}{B}=0$; the angle between the line and the x-axis is zero, and tan $0=0$.

When $B=0$; then $Ax+C=0$ gives $x=-\frac{C}{A}$, representing a line parallel to the y-axis, distant $-\frac{C}{A}$ units from it. Here $m=-\frac{A}{B}=\infty$; the angle between the line and the x-axis is 90°, and tan $90°=\infty$.

$x=0$; this places no restriction on y, but requires x to be zero; it is therefore the equation of the y-axis.

$y=0$ is similarly the equation of the x-axis.

100. Tracing a Straight Line.—A straight line is determined if two of its points are given, or if one point and its direction are given; there are thus two simple ways of tracing straight lines. For instance, to trace the locus of $3x-5y+6=0$. When $y=0$, $x=-2$; the point $(-2, 0)$ is on the line. When $x=0$, $y=1.2$; the point $(0, 1.2)$ is on the line. Plotting these intersections of the line with the axes, and drawing a straight line through them, we have the required locus. (See Fig. 9a.)

Again, we may solve, getting $y=\frac{3}{5}x+\frac{6}{5}$. Mark a point on the y-axis $\frac{6}{5}$ units above the origin; thence draw a line of any length parallel to the x-axis, and at its extremity erect a per-

pendicular $\frac{3}{5}$ as long; the hypothenuse completing the right triangle will, if produced, be the required locus. (See Fig. 9b.)

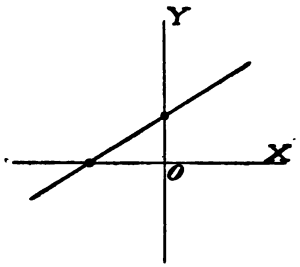


FIG. 9a.

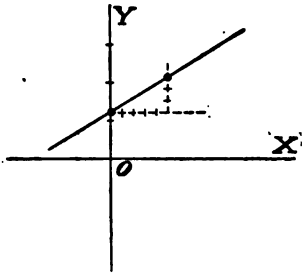


FIG. 9b.

The distances from the origin to the points where a straight line cuts the x-axis and the y-axis are called the *x-intercept* and the *y-intercept* of the line; in the example, they are -2 and $\frac{6}{5}$ respectively.

Instead of using the two intercepts or the y-intercept and the slope, we can find a point of a line by giving to x or y in the equation any value, and computing the value of the other co-ordinate, thus finding the coordinates of a point of the line. Then the direction-ratio (slope) may be used as before, or a second point found.

Examples.

Trace the following:

1. $y=3x+2$.
2. $y=-2x-3$.
3. $2x+y-7=0$.
4. $3x-5y+9=0$.
5. $x-4=0$.
6. $y+2=0$.
7. $x=0$.
8. $y=0$.
9. $10x+y=50$.
10. $x-15y+90=0$.
11. $x=y$.
12. $y=-x$.
13. $x=2y$.
14. $4y-x=0$.
15. $3x+5y=0$.
16. $y=-\frac{1}{3}x$.

101. Finding the Equation of a Straight Line.—In general formulæ, the coordinates of a given point are represented by (x_1, y_1) or (x_2, y_2), etc., the subscripts indicating that the values are known constants.

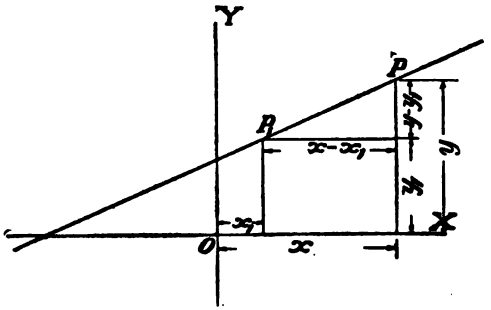


FIG. 10.

To find the equation of a straight line passing through a given point (x_1, y_1) and having a given slope m. Let P_1 (x_1, y_1) be the given point, and P (x, y) be any other point of the line (Fig. 10). Complete the figure as shown; then, by hypothesis (Art. 80),

$$\frac{y-y_1}{x-x_1}=m,$$

which is the desired equation.

The equation of the straight line through the point (x_1, y_1) *and having the slope* m *is* $y-y_1=m(x-x_1)$.

If the given point is $(0, b)$, so that the line is given by its *y-intercept and its slope,* we have the familiar equation

$$y-b=m(x-0) \text{ or } y=mx+b,$$

where b is the y-intercept.

If the given point is the *origin* $(0, 0)$, the equation is $y=mx$.

If $m=\infty$, we have $\frac{x-x_1}{y-y_1}=\frac{1}{m}=0$, $x=x_1$, which represents a straight line parallel to the y-axis, through $x=x_1$.

The equation of the straight line through the points (x_1, y_1) *and* (x_2, y_2) *is*

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1);$$

for, as is evident from Fig. 11, $\frac{y_2 - y_1}{x_2 - x_1}$ is m, the slope.

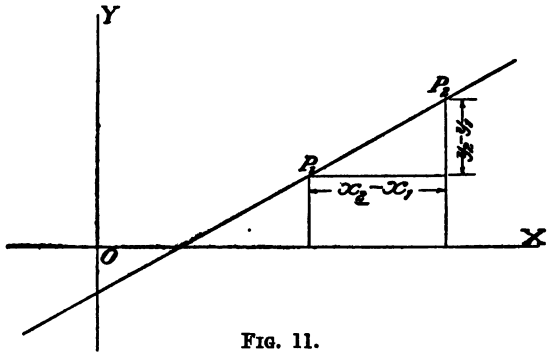


FIG. 11.

If the x-intercept is given $= a$, the y-intercept $= b$, the slope is seen by Fig. 12 to be $-\frac{b}{a}$ (the negative sign indicating a slope downward from left to right. See Arts. 83 and 99). Hence the equation is

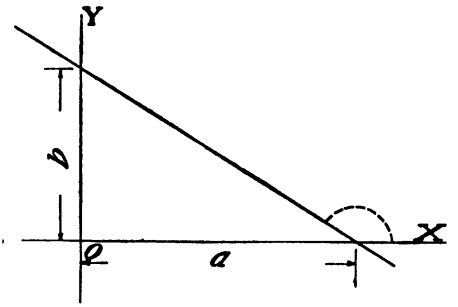


FIG. 12.

$$y = -\frac{b}{a}x + b;$$

or, dividing by b and transposing,

$$\frac{x}{a} + \frac{y}{b} = 1$$

is the equation of a straight line in terms of its intercepts.

Examples.

Find or write the equations of the following straight lines:

1. Through $(2, -3)$ with slope $-\frac{1}{2}$. *Ans.* $x+2y+4=0$.
2. Through $(-1, 2)$ with slope 3. *Ans.* $3x-y+5=0$.
3. Through $(1, 3)$ with slope 0, and again, with slope ∞. *Ans.* $y=3$ and $x=1$.
4. Through $(-1, -1)$ and $(3, 2)$. *Ans.* $3x-4y-1=0$.
5. Through $(1, 0)$ and $(3, 1)$. *Ans.* $x-2y-1=0$.
6. Through $(-1, 2)$ and $(2, -1)$. *Ans.* $x+y-1=0$.
7. x-intercept $-\frac{4}{3}$, y-intercept $\frac{1}{3}$. *Ans.* $9x-12y+4=0$.
8. x-intercept 2, y-intercept -3. *Ans.* $3x-2y-6=0$.

102. Parallels and Perpendiculars.—It is evident that *two parallel straight lines have the same direction-ratio or slope,* and that conversely *two equations* $y=mx+b_1$ and $y=mx+b_2$ represent parallel lines.

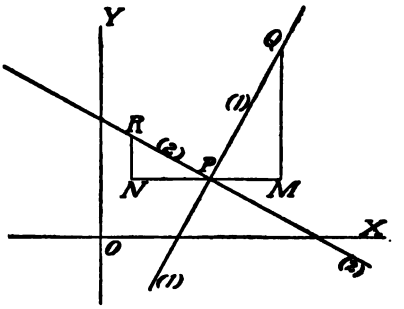


FIG. 13.

Suppose we have any two perpendicular lines, as shown in Fig. 13. Let P be their intersection, and draw the right triangles as shown. The triangles are similar, since their sides are respectively perpendicular, so that

$$\frac{NR}{NP}=\frac{PM}{MQ}.$$

If the slope of PQ is m_1, and that of PR is m_2,

$$m_2 = -\frac{NR}{NP} = -\frac{PM}{MQ} = -\frac{1}{m_1}.$$

The steps of this proof can be retraced; so that:

Two perpendicular straight lines have direction-ratios or slopes that are negative reciprocals of each other; and conversely, *two equations* $y = mx + b_1$ *and* $y = -\frac{x}{m} + b_2$ *represent perpendicular lines.*

Briefly, $y = m_1x + b_1$ *and* $y = m_2x + b_2$ *are parallel if* $m_2 = m_1$, *perpendicular if* $m_2 = -\frac{1}{m_1}$.

103. Equation of a Line through a Given Point and Parallel or Perpendicular to a Given Line.—To find the equation of a straight line through $(2, -3)$ and parallel or perpendicular to $3x - 7y + 2 = 0$ we proceed as follows:

The equation of the given line may be written

$$y = \tfrac{3}{7}x + \tfrac{2}{7};$$

hence the slope is $\frac{3}{7}$. The slope of any parallel line is $\frac{3}{7}$, of any perpendicular line, $-\frac{7}{3}$.

For the parallel through $(2, -3)$ we have (Art. 101)

$$y + 3 = \tfrac{3}{7}(x - 2) \quad \text{or} \quad 3x - 7y - 27 = 0.$$

For the perpendicular through $(2, -3)$ we have

$$y + 3 = -\tfrac{7}{3}(x - 2) \quad \text{or} \quad 7x + 3y - 5 = 0.$$

The method is evidently general.

Examples.

Find the equations of the following lines:

1. A parallel to $3y-2x+4=0$ through $(-2, 3)$.
   Ans. $3y-2x=13$.
2. A parallel to $3y+5x-1=0$ through $(3, 0)$.
3. A parallel to $4y+x-7=0$ through $(0, 0)$.
4. A perpendicular to $7x+3y+5=0$ through $(3, -4)$.
   Ans. $3x-7y=37$.
5. A perpendicular to $5x-y+1=0$ through $(0, -2)$.
6. A perpendicular to $2x+3y-11=0$ through $(0, 0)$.
7. Prove that the parallel and the perpendicular to
$$ax+by+c=0$$
through (x_1, y_1) are $ax+by=ax_1+by_1$ and $bx-ay=bx_1-ay_1$, respectively.

104. The Mid-Point.—The point half-way between (x_1, y_1) and (x_2, y_2) is
$$\left(\frac{(x_1+x_2)}{2}, \frac{y_1+y_2}{2}\right).$$

This appears from the similar triangles in Fig. 14, since the coordinates of the mid-point are

$$x_1+\frac{x_2-x_1}{2} \quad \text{and} \quad y_1+\frac{y_2-y_1}{2}.$$

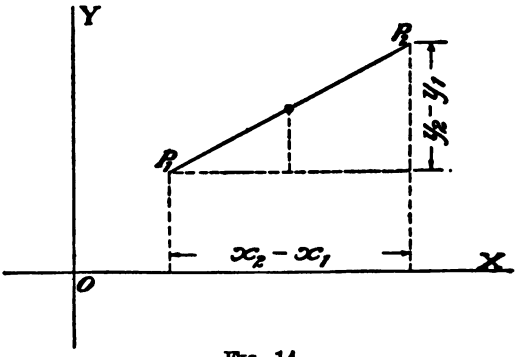


FIG. 14.

Coordinates can be computed in this way for a point dividing the distance between two given points in any given ratio.

105. The Distance between Two Points.—As the legs of the right triangle in Fig. 14 are (x_2-x_1) and (y_2-y_1), the distance between (x_1, y_1) and (x_2, y_2) is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

106. The Distance from a Point to a Line.—To find the distance from any point $P(x_1, y_1)$ to any straight line $y=mx+b$. In Fig. 15, let BM be the given line and P the given point, and $PM=d$ the required distance.

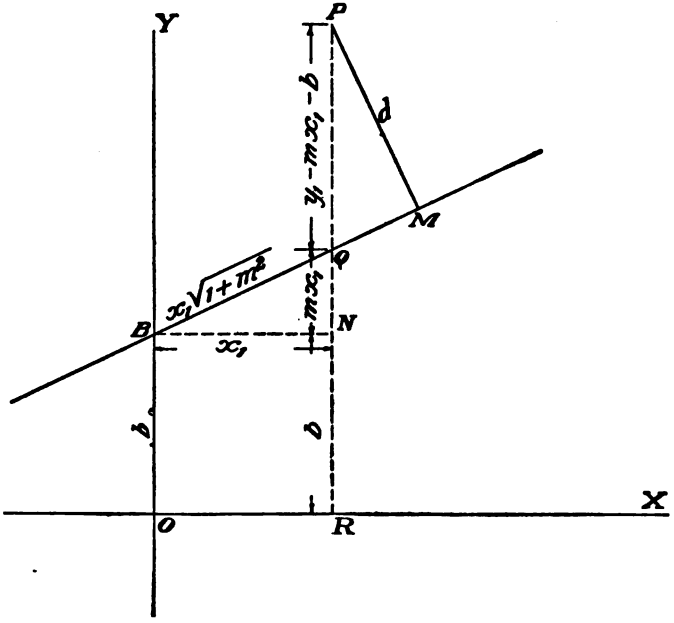


FIG. 15.

$$\frac{NQ}{BN}=m, \quad \text{and} \quad BN=x_1;$$

$$\therefore\ NQ=mx_1.$$

$$RN=OB=b,\ RP=y_1;$$

$$\therefore\ QP=y_1-mx_1-b.$$

$$QB=\sqrt{x_1^2+(mx_1)^2}=x_1\sqrt{1+m^2}.$$

The $\triangle\ PMQ$ is similar to the $\triangle\ BNQ$; hence

$$\frac{PM=d}{QP=y_1-mx_1-b}=\frac{BN=x_1}{QB=x_1\sqrt{1+m^2}}=\frac{1}{\sqrt{1+m^2}}.$$

Then

$$d=\frac{y_1-mx_1-b}{\sqrt{1+m^2}}.$$

When the point (x_1, y_1) is on one side of BM, y_1-mx_1-b is negative; when (x_1, y_1) is on the other side, y_1-mx_1-b is positive; when (x_1, y_1) is on BM, y_1-mx_1-b is zero. Only the numerical value of d is important.

The distance from (x_1, y_1) to $Ax+By+C=0$, or to

$$y=-\frac{A}{B}x-\frac{C}{B},$$

is

$$d=\frac{y_1+\frac{A}{B}x_1+\frac{C}{B}}{\sqrt{1+\frac{A^2}{B^2}}}=\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

For instance, required the distance from $(4, -3)$ to

$$2y-5x=8;$$

rewrite the equation $5x-2y+8=0$; then

$$d=\frac{5\times4-2(-3)+8}{\sqrt{4+25}}=\frac{34}{\sqrt{29}}.$$

Examples.

1. Find the equation of the mid-perpendicular to the line joining $(-1, 3)$ and $(3, -5)$.

2. Find the following distances: From $(-3, 7)$ to $(0, 3)$; from $(-2, -5)$ to $(3, 7)$. *Ans.* 5 and 13.

3. The vertices of a triangle being $A(-2, 0)$, $B(2, 0)$, and $C(1, 5)$, find the middle point of each side, and the lengths of the lines joining the mid-points. Check by showing them to be one-half as long as the sides.

4. Find the equations of AC and BC in Example 3 and the lengths of the altitudes of the triangle perpendicular to them.

5. Find the distance of $(-2, 1)$ from $3x-4y=5$. *Ans.* 3.

6. Find the distance of $(1, -3)$ from $12y=5x+11$.

107. Intersections of Lines; Simultaneous Solutions.—Suppose we have given the equations of two lines:

$$AB: \quad 3x-y+2=0,$$

$$CD: \quad x+2y-6=0,$$

and wish to find the coordinates of their common point, P. (See Fig. 16.)

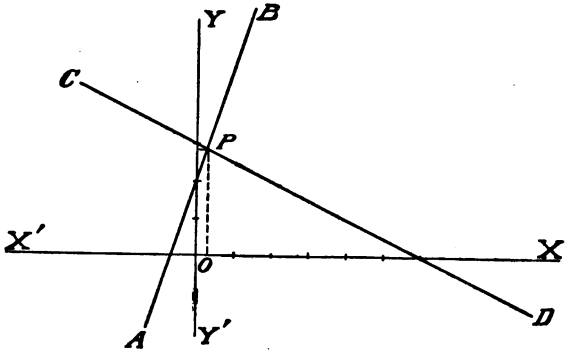


FIG. 16.

The coordinates of P satisfy the equation of AB, because P is on AB; and they satisfy the equation of CD, because P is on CD. As they thus satisfy both equations, they can be obtained by solving the two equations simultaneously. Solving, we get $x=\frac{3}{7}$, $y=\frac{20}{7}$, so that P is the point $(\frac{3}{7}, 2\frac{6}{7})$.

This figure may evidently be used to represent graphically the common solution of the two simultaneous equations.

In general:

The coordinates of any common point of two curves may be found by the simultaneous solution of the equations of the curves, and:

Any pair of values of x and y satisfying two simultaneous equations in x and y are the x and y coordinates of a point common to the graphs of the equations.

108. A special case is encountered if we attempt to solve two such equations as

$$2x-3y=7,$$
$$2x-3y=8;$$

for any attempt at elimination of one variable, leads to the elimination of both, with an absurd result, $1=0$. We have seen that such a result indicates that the root to be expected has disappeared through becoming infinite. This is consistent, as the two lines are parallel, and form the limiting case of lines meeting at an increasingly great distance from the origin. For instance,

$2x-\ 3.01y=7$ and $2x-3y=8$ meet at (154, 100),

$2x-3.0001y=7$ and $2x-3y=8$ meet at (15004, 10000),

and so on.

This relation is expressed by saying that *parallel lines meet at infinity.*

This appears also in another way. If (x_1, y_1) is any point on the line $y=mx+b$, the ratio of the coordinates $\frac{y_1}{x_1}$ is

$$\frac{y_1}{x_1} = m + \frac{b}{x_1},$$

and as x_1 (and therefore y_1 also) increases indefinitely,

$$\frac{y_1}{x_1}\Big]_{x_1=y_1=\infty} = m + \frac{b}{x_1}\Big]_{x_1=\infty} = m,$$

the ratio of y to x at ∞ thus being independent of b.

Examples.

Show graphically and estimate the simultaneous solutions of the following pairs of equations:

1. $3y-2x=5$ and $6y-x=3$.
2. $5y+12x=8$ and $4y-10x=7$.
3. $5y-12x=4$ and $3y-7x=6$. (Use a small scale.)

109. Finding Geometric Loci by Algebraic Processes.—To find the locus of points equidistant from two given points, A and B. Call the distance $AB=2a$; take AB as x-axis and the mid-perpendicular to AB as y-axis. Then (see Fig. 17) the coordinates of A are $(-a, 0)$; of B, $(a, 0)$. Let the coordinates

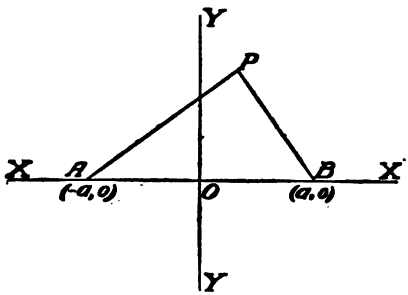


FIG. 17.

of any point P of the desired locus be (x, y). Then, by hypothesis, the distance PA = the distance PB, or

$$\sqrt{[x-(-a)]^2+(y-0)^2}=\sqrt{(x-a)^2+(y-0)^2};$$

whence, squaring, collecting terms and simplifying,

$$(x+a)^2+y^2=(x-a)^2+y^2,$$
$$4ax=0;$$
$$x=0.$$

We find that the coordinates (x, y) of P are subject to the restriction $x=0$; this is the equation of the desired locus. The locus is thus seen to be the y-axis, or the mid-perpendicular to AB.

This problem is typical, and the steps taken in solving it should be carefully noted:

To find the locus of a point whose motion is restricted by a given condition:

(1) Choose two convenient perpendicular lines as axes.

(2) Locate every given point by coordinates, to be treated as known constants.

(3) Represent the variable coordinates of any point of the locus by (x, y).

(4) Put the given condition in the form of an equation involving x and y.

(5) Simplify this equation, which is the equation of the required locus.

(6) Identify the locus from its equation.

110. The Equation of the Circle.—To find the equation of a circle with given radius a. First, let the center of the circle be at the origin (Fig. 18), and let $P(x, y)$ be any point of the circle.

The distance $OP=\sqrt{x^2+y^2}$ is equal to a, by the definition of the circle.

$$x^2+y^2=a^2$$

is therefore *the equation of a circle with the radius a having its center at the origin.*

Again, let the center of the circle be at the given point (b, c), and let $P(x, y)$ be any point of the circle.

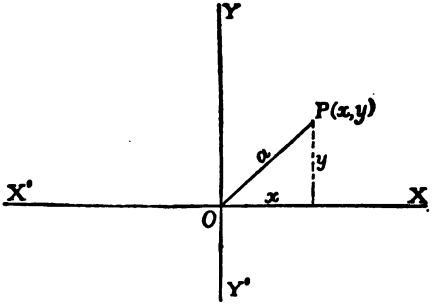


FIG. 18.

The distance from (x, y) to (b, c) is a; hence

$$\sqrt{(x-b)^2+(y-c)^2}=a \quad \text{or} \quad (x-b)^2+(y-c)^2=a^2$$

is the equation of a circle with the radius a and the center (b, c).

This equation, when simplified, is

$$x^2+y^2-2bx-2cy-a^2+b^2+c^2=0;$$

it has the peculiarities that it is of the second degree, but has no xy term, and that the coefficients of x^2 and y^2 are equal.

Any equation having these peculiarities is the equation of a circle, and its locus is easily identified. For instance,

$$2x^2+2y^2-6x+10y+9=0,$$

by adding to both sides enough to complete the two squares, may be written in the typical form as

$$(x-\tfrac{3}{2})^2+(y+\tfrac{5}{2})^2=4;$$

hence the equation represents a circle with the radius 2 and the center $(\frac{3}{2}, -\frac{5}{2})$.

Examples.

1. Find the locus of points equally distant from the axes of coordinates.

2. Find the locus of points equally distant from $2x-3y+5=0$ and $3x-2y+6=0$.

Ans. $\pm(2x-3y+5)=\pm(3x-2y+6)$; $x+y+1=0$
or $5x-5y+11=0$.

3. Find the locus of points five units from $(-2, 1)$.

4. Find the locus of points equidistant from $(-3, 1)$ and $(3, 3)$ by the process of Art. 109, and check by finding the equation of the mid-perpendicular to the line joining the given points.

5. The base of a triangle is the part of the x-axis from $(-3, 0)$ to $(2, 0)$, and the bisector of its vertical angle passes through the origin. Find the equation of the locus of the vertex, and identify it. (Use the theorem about the segments into which the base of a triangle is divided by the bisector of the vertical angle.)

6. Locate the center and give the radius of
$x^2+y^2-12x+10y+12=0$; $x^2+y^2+8x-2y-8=0$.

7. Find the equation of a circle having the center $(1, -1)$ and passing through $(3, 2)$.

111. Simultaneous Equations.—We can now represent graphically the simultaneous solution of any linear equation and a quadratic equation of the type

$$ax^2+ay^2+dx+ey+f=0,$$

or of two such quadratics.

These solutions correspond, as we have seen, to the coordinates of the intersection of the loci of the two equations. The real roots can be approximated by measuring the coordinates of the intersections, and the figure will show the nature of the roots. Two pairs of roots are to be expected; if the loci do not intersect, these roots are imaginary, and if the loci are tangent, so that two intersections have merged, the two pairs of roots will be alike.

The quadratic terms can be eliminated from any pair of equations of the type

$$ax^2+ay^2+dx+ey+f=0,$$

and if we solve the resulting linear equation with either of the given quadratic equations, we shall get the simultaneous solutions of the two quadratics.

For instance, given

$$x^2+y^2+12x+8y+35=0 \qquad (1)$$

and

$$x^2+y^2-8x-4y-65=0. \qquad (2)$$

Subtracting, we have

$$20x+12y+100=0,$$

or

$$5x+3y+25=0. \qquad (3)$$

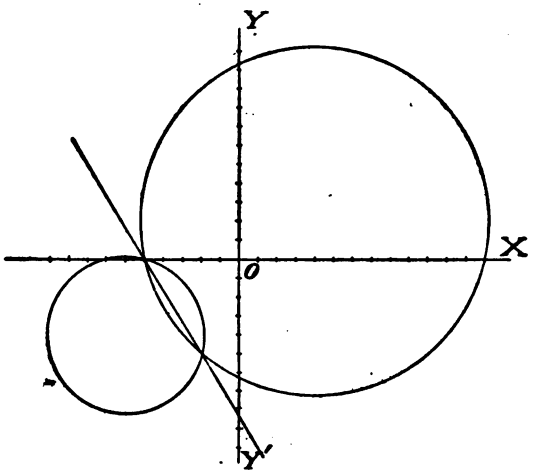


FIG. 19.

(1) and (2) represent circles and (3) represents a straight line. Any point satisfying both (1) and (2) must also satisfy (3); but any point satisfying either (1) or (2) and not the other cannot satisfy (3). Hence the locus of (3) meets the loci of (1) and (2) at their common points, but not anywhere else, and so is the common chord of the two circles. (See Fig. 19.)

112. The principle brought out here is general, and may be stated:

k and l being constant, $kf(x, y) - l\phi(x, y) = 0$ represents a curve passing through all the common points of $f(x, y) = 0$ and $\phi(x, y) = 0$, and not meeting either at any other point; for a point common to both makes $f(x, y)$ and $\phi(x, y)$ both zero, and a point that is on one and not on the other, makes either $f(x, y)$ or $\phi(x, y)$ zero, but not both.

A related principle is:

$f(x, y) \times \phi(x, y) = 0$ represents the two curves, $f(x, y) = 0$ and $\phi(x, y) = 0$, together; for any point of either curve makes $f(x, y)$ or $\phi(x, y)$ equal to zero and so makes the product zero.

Thus $(2x-3y+1)(3x+y-2) \equiv 6x^2-7xy-3y^2-x+7y-2=0$ represents two straight lines; $xy=0$ represents the coordinate axes; $x^2-y^2=0$ represents the bisectors of the angles between the axes.

Examples.

Illustrate by graphs the solution of the following simultaneous equations (check by solving and plotting the solutions):

1. $x^2+y^2+6x-6y-47=0$ with each of the following in turn (draw one circle cut by the various lines):

   (a) $y-x+5=0$, (b) $y-x+1=0$, (c) $y+x=0$,
   (d) $3y-8x+28=0$, (e) $5x-y=21$, (f) $8x+y=44$.

2. $x^2+y^2-10x-8y-24=0$ with $y+x=0$.
3. $x^2+y^2-10x-8y-24=0$ with $y^2-x^2=0$.
4. $x^2+y^2+6x-6y-47=0$ with $x^2+y^2-6x-11=0$.
   $x^2+y^2+6x-6y-47=0$ with $x^2+y^2-10x+2y+1=0$.
   $x^2+y^2+6x-6y-47=0$ with $x^2+y^2-14x+4y+13=0$.
5. $x^2+y^2-2x-4y=0$ with $x^2+y^2-14x+2y-30=0$.

113. Derivatives of y^2 and xy.—When the derivatives of more than one function are used at the same time, we are obliged to add to our notation. As we saw in the discussion of Art. 84, the derivative with respect to x of a given function of x is the slope of the tangent to the graph of the function, and is determined by finding the limit of the slope of a secant line as it swings into coincidence with the tangent. If the function

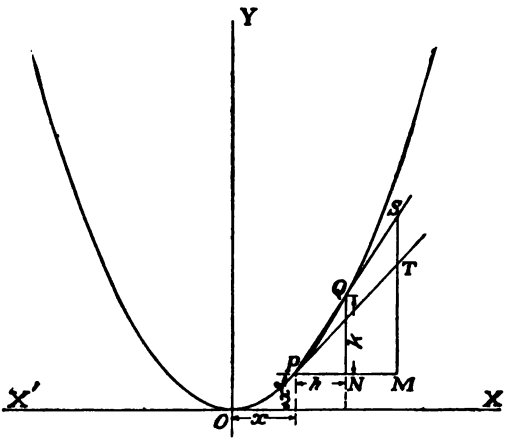


FIG. 20.

treated is represented by y, the increase given to x (PM, Fig. 20) is represented by dx, a compound symbol meaning "*differential of x*"; the length MT, the height of the tangent line above the point M, is represented by dy, meaning "*differential of y.*" Thus the derivative of y with respect to x, defined as the value of $\frac{MT}{PM}$, is represented by the symbol $\frac{dy}{dx}$, which is ordinarily read either "*dy over dx*" or "*the derivative of y with respect to x.*" In other words, *when* $f(x)$ *is represented by* y, $f'(x)$ *is represented by* $\frac{dy}{dx}$.

If y is a function of x, y^2 and xy are also functions of x; their derivatives are represented by $\frac{d(y^2)}{dx}$ and $\frac{d(xy)}{dx}$. We shall see that each of these derivatives bears a useful relation to the derivative $\frac{dy}{dx}$.

Suppose x is increased (see Fig. 20) to the value $(x+h)$; as y is a function of x, its value must change correspondingly. Call its new value $(y+k)$; then y^2 also changes to the value $(y+k)^2$. There are thus three corresponding increases:

The increase in x is h.

The increase in y is k.

The increase in y^2 is $(y+k)^2-y^2=2yk+k^2$.

By definition:

The derivative of y with respect to x is

$$\frac{dy}{dx}=\left[\frac{k}{h}\right]_{h=0}.$$

The derivative of y^2 with respect to x is

$$\frac{d(y^2)}{dx}=\left[\frac{2yk+k^2}{h}\right]_{h=0}.$$

Then

$$\frac{d(y^2)}{dx}=\left[2y\frac{k}{h}+k\frac{k}{h}\right]_{h=0}=2y\frac{dy}{dx}+0\cdot\frac{dy}{dx}=2y\frac{dy}{dx},$$

since $k=0$ when $h=0$.

Hence, *if y is a function of x, the derivative of y^2 with respect to x is $2y$ times the derivative of y with respect to x;* or

$$\frac{d(y^2)}{dx}=2y\cdot\frac{dy}{dx}.$$

In the same way, as x changes to $(x+h)$ and y to $(y+k)$, xy changes to $(x+h)(y+k)$, receiving the increase

$$(x+h)(y+k)-xy=yh+xk+hk.$$

Again, by definition:

The derivative of xy with respect to x is

$$\frac{d(xy)}{dx}=\left[\frac{yh+xk+hk}{h}\right]_{h=0}.$$

Then

$$\frac{d(xy)}{dx}=\left[y+x\frac{k}{h}+k\right]_{h=0}=y+x\cdot\frac{dy}{dx}.$$

Hence, *if y is a function of x, the derivative of xy with respect to x is y plus x times the derivative of y;* or

$$\frac{d(xy)}{dx}=y+x\frac{dy}{dx}.$$

114. Derivative of an Implicit Function.—The equation $x^2+y^2=a^2$ defines implicitly the function $y=\pm\sqrt{a^2-x^2}$; the graph of y, or the locus of $x^2+y^2=a^2$, is a circle, as we know. Now although in the equation $x^2+y^2=a^2$, x and y may vary, the quantity (x^2+y^2) will be *constantly* a^2 as long as x and y are coordinates of a point of the circle; hence the derivative of x^2+y^2 must be zero if the point (x, y) is on the circle. Then

$$2x+2y\frac{dy}{dx}=0$$

for a point on the circle, and

$$\frac{dy}{dx}=-\frac{x}{y}.$$

If (x, y) is a point on the circle, the slope of the circle at the point (x, y) is $-\frac{x}{y}$; that is, if we put for x and y the coordi-

nates of some point on the circle, the resulting value of $-\frac{x}{y}$ will be the slope at that point.

A similar discussion to this given for the circle will show for any curve that:

The general expression for the slope, $\frac{dy}{dx}$, of any curve at any point of the curve, may be determined by taking the derivatives of both sides of the equation of the curve and solving the result for $\frac{dy}{dx}$.

$\frac{dy}{dx}$ will thus be given in terms of x or y or both x and y, and it must be observed that the expression is useless until we have substituted in it the coordinates of some definite point of the curve, thus getting the slope of the curve, or of its tangent, *at that point.*

115. Tangents to Circles.—Let it be required to find the tangent to the circle $x^2+y^2-3x+5y+6=0$ at the point $(1, -4)$. This is possible by the method just described; for $(1, -4)$ lies on the given circle, since putting its coordinates in the equation gives

$$1+16-3-20+6=0.$$

Taking derivatives,

$$2x+2y\frac{dy}{dx}-3+5\frac{dy}{dx}=0.$$

Solving,

$$\frac{dy}{dx}=\frac{3-2x}{2y+5}.$$

Putting $x=1$, $y=-4$,

$$\left.\frac{dy}{dx}\right]_{1,-4}=\frac{3-2}{-8+5}=-\tfrac{1}{3}.$$

We now know one point $(1, -4)$ of the tangent, and its slope, $-\frac{1}{3}$, and can draw the tangent. We can also write its equation,

$$y+4=-\tfrac{1}{3}(x-1), \quad \text{or} \quad x+3y+11=0.$$

116. Equation of Tangent to $x^2+y^2=a^2$ at Any Point (x_1, y_1).—Let $P_1(x_1, y_1)$ be any point of the circle $x^2+y^2=a^2$, so that $x_1^2+y_1^2=a^2$; then the slope of the tangent to the circle at P_1 is $-\frac{x_1}{y_1}$, and the equation of the tangent is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1), \text{ or } y_1y-y_1^2=-x_1x+x_1^2;$$

this is,

$$x_1x+y_1y=x_1^2+y_1^2=a^2,$$

since $x_1^2+y_1^2=a^2$ by hypothesis.

The tangent to $x^2+y^2=a^2$ at any point (x_1, y_1) of the curve is $x_1x+y_1y=a^2$.

Examples.

Find the equation of the tangent to each of the following circles at the given point; then trace the circle and the tangent independently. Check your result by the figure and by finding the intersection of the circle and the tangent.

1. $x^2+y^2-4x-2y-15=0$ at $(-2, 3)$.
2. $x^2+y^2-2x+6y+5=0$ at $(2, -1)$.
3. $x^2+y^2+6x-4y+4=0$ at $(-3, -1)$.
4. $x^2+y^2+2x-6y+1=0$ at $(2, 3)$.
5. Show by comparing the direction-ratio of the radius to a point (x_1, y_1) of the circle $x^2+y^2=a^2$ with that of the tangent at this point, that the lines are perpendicular.
6. Show that the tangent to the graph $y=ax^3+bx^2+cx+d$ at the point where the graph crosses the axis of y is the line $y=cx+d$. What bearing has this on the determination of figures of the root in Horner's Method?

117. Tangent to Circle in Terms of its Slope.—Given that a line $y=mx+b$ having a given direction is tangent to the circle $x^2+y^2=a^2$, m thus being determined, what must be the value of b?

The distance of the line from the center $(0, 0)$ of the circle is

$$\left.\frac{mx-y+b}{\pm\sqrt{1+m^2}}\right]_{(0,0)}=\frac{\pm b}{\sqrt{1+m^2}},$$

and is equal to a if the line is a tangent. Hence

$$b=\pm a\sqrt{1+m^2}.$$

Consequently, *for any value of* m,

$$y=mx\pm a\sqrt{1+m^2}$$

is tangent to $x^2+y^2=a^2$.

The double sign corresponds to the two tangents possible with a given slope.

In the same way, if $y=mx+b$ is tangent to any circle $(x-c)^2+(y-d)^2=a^2$, the distance from the center (c, d) to the tangent is

$$\frac{mc-d+b}{\sqrt{1+m^2}}=\pm a,$$

whence

$$b=d-mc\pm a\sqrt{1+m^2},$$

and the tangent is

$$y-d=m(x-c)\pm a\sqrt{1+m^2}.$$

118. The Tangent to the Circle from an Outside Point.—Let it be required to find the tangents to $x^2+y^2+4x-8y-5=0$ from the point $(2, -1)$.

The circle is $(x+2)^2+(y-4)^2=25$, so that any tangent to it having the slope m is

$$y-4=m(x+2)\pm 5\sqrt{1+m^2}.$$

The point $(2, -1)$ lies on the required tangent, so that if this is to be its equation we must have

$$-1-4=m(2+2)\pm 5\sqrt{1+m^2} \quad \text{or} \quad -5-4m=\pm 5\sqrt{1+m^2}.$$

Squaring and collecting terms,

$$9m^2-40m=0;$$

so that for the required tangent $m=0$ or $\frac{40}{9}$. The two tangents are thus

$$y=-1$$

and

$$y+1=\tfrac{40}{9}(x-2) \quad \text{or} \quad 40x-9y-89=0.$$

Examples.

1. Find the tangents to $x^2+y^2+8x-10y+1=0$ having the slope $-\frac{1}{3}$; trace the circle and the tangents.

2. Find the tangents to the circle of Example 1 from the point $(3, -4)$; trace the circle and the tangents.

3. Find the tangents to $x^2 + y^2 - 6x - 10y - 18 = 0$ from $(-1,-2)$; trace the circle and the tangents.

4. Trace in one figure the lines represented by

$$y-2=m(x-1)\pm 2\sqrt{1+m^2}$$

when $m=-\frac{12}{5}$, $-\frac{4}{3}$, $-\frac{3}{4}$, $-\frac{5}{12}$, 0, $\frac{5}{12}$, $\frac{3}{4}$, $\frac{4}{3}$, $\frac{12}{5}$, respectively, and draw the circle that is tangent to all of them.

119. Loci of Quadratic Equations.—The most general form of quadratic equation in two variables is

$$ax^2+bxy+cy^2+dx+ey+f=0. \qquad (1)$$

If some particular value x_1 is given to x, such an equation becomes a quadratic in one unknown quantity, y, which can be solved. This is equivalent to finding the intersection of the locus of (1) with the line $x=x_1$; generally two points in this vertical line are found, but they may be imaginary, indicating that the line does not cut the locus, or coincident, indicating that the line is tangent to the locus.

Enough points to plot the locus might be found in this way, but the process would be very tedious. It is much simpler to solve Equation (1) for y in terms of x once for all, and substitute various values of x in the result. In this process, certain principles become evident which shorten the work.

The procedure varies according as the factors of the terms of highest degree, $ax^2+bxy+cy^2$, are imaginary, real and equal, or real and different; that is, according to whether b^2-4ac is less than zero, equal to zero, or greater than zero. The loci in these three cases are given the names of *ellipse, parabola,* and *hyperbola,* respectively.

120. The Ellipse.—Given the equation

$$2x^2-2xy+y^2+2x-4y+1=0, \qquad (1)$$

for which $b^2-4ac=-4$, to find its locus.

First solve for y, thus expressing y as an explicit function of x:

$$y=x+2\pm\sqrt{(3-x)(x+1)}. \qquad (2)$$

First trace the line (Fig. 21)

$$y=x+2; \qquad (3)$$

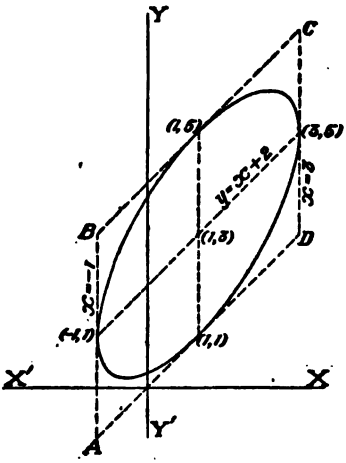


FIG. 21.

for any value we may give to x, we get one point on this line, and two points on the graph of (2), one $\sqrt{(3-x)(x+1)}$ vertically above the point on (3), the other $\sqrt{(3-x)(x+1)}$ vertically below. The curve of (2) consequently has a vertical chord $2\sqrt{(3-x)(x+1)}$ in length, bisected by $y=x+2$. As this is true for every value of x, the line $y=x+2$ bisects every vertical chord of (2), and so is called a *diameter* of the ellipse.

It is evident that we shall get real values of y from (2) only when x is less than 3 and greater than -1; there is therefore no part of the curve to the right of the vertical line $x=3$, or to the left of the vertical line $x=-1$. Moreover, when $x=3$ or -1, the two values of y become coincident, and the length of the vertical chord becomes zero; i. e., $x=3$ and $x=-1$ are tangent to the curve. They are called the *vertical limiting tangents.*

The length of the vertical chord, $2\sqrt{(3-x)(x+1)}$, has its greatest value when the value of x is half-way between the limiting values, 3 and -1, or when $x=1$. The extremities of this maximum vertical chord have the coordinates

$$x=1 \quad \text{and} \quad y=1+2\pm\sqrt{(3-1)(1+1)}=3\pm 2=5 \text{ or } 1;$$

that is, are the points (1, 5) and (1, 1), the length of the chord being 4.

It can be shown that the vertical line $x=1$ is another diameter of the ellipse, bisecting all chords parallel to $y=x+2$; the two diameters are said to be conjugate. Their intersection is the *center* of the ellipse, the point (1, 3).

It can be shown that parallels to $y=x+2$ (the conjugate to the vertical diameter) drawn through the ends of the vertical diameter are tangent to the ellipse.

We have thus located a parallelogram such that the ellipse is tangent to each of its sides at the middle point. This parallelogram determines the ellipse completely, and is sufficient for what is called *tracing* the ellipse; i. e., sketching a smooth oval having the properties so far described.

121. Tracing an Ellipse.—The steps of the process may be summarized:

Given any equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ having $B^2-4AC<0$:

(1) *Solve* for y.

(2) Trace the *oblique diameter.*

(3) Draw the *vertical limiting tangents.*

(4) Draw the *vertical diameter,* locate its *extremities,* determine its *length,* and locate the *center.*

(5) Draw *parallels* to the oblique diameter through the extremities of the vertical diameter.

(6) *Sketch the ellipse,* tangent to the parallelogram at the ends of the two conjugate diameters.

It is possible to solve similarly for x in terms of y and get horizontal limiting tangents, the horizontal diameter and the diameter conjugate to it.

122. Construction of an Ellipse.—It can be proved that the following construction will give points of an ellipse.

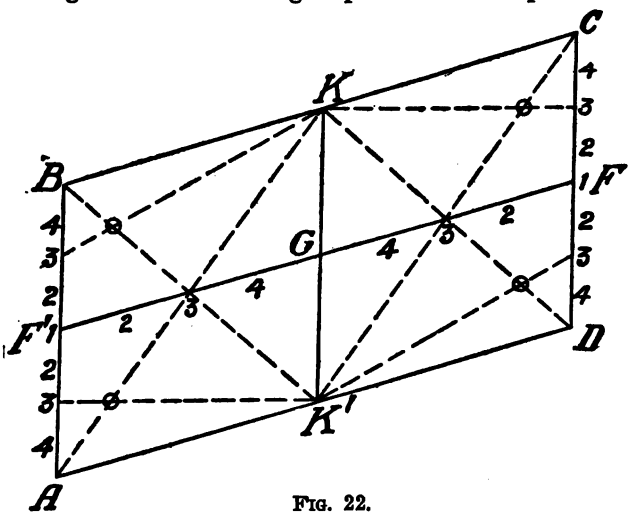


FIG. 22.

Given a parallelogram $ABCD$, Fig. 22, divided into four parallelograms by lines $F'GF$, $K'GK$ drawn through its center, G, parallel to its sides. Divide the line FG into any number of equal parts, and number the points of division consecutively, beginning with F. Divide FC into the *same* number of equal parts and number the points of division, beginning at F.

Join K with any point of FC, and K' with the corresponding point of FG, the two lines intersecting at P; then P is a point of the ellipse tangent to the sides of the parallelogram at F', K, F and K'.

Treating the other corners of the parallelogram in the same way, we can find any number of points of the ellipse. The four points shown in the figure are especially useful and very easily found.

Examples.

1. Trace $4y^2-4xy+5x^2-8y+12x-28=0$, solving for y.

2. Trace $16y^2+24xy+25x^2-96y-136x+64=0$, solving for y and x both.

3. For any point on the ellipse of Example 1,

$$\frac{dy}{dx}=\frac{2y-5x-6}{2(2y-x-2)}.$$

What is the slope of the ellipse at its intersections with $2y-5x-6=0$? With $2y-x-2=0$? What are these lines? What is their intersection?

4. Find the locus of points $\frac{1}{2}$ as far from the point $(2, 2)$ as from the line $x+y=0$, and trace it.

5. In the ellipses of Examples 1 and 2, find the slope of the tangents at the ends of the vertical diameter, showing it to be the same as the slope of the diameter conjugate to the vertical.

6. Given the ellipse $3x^2-xy+y^2+6x+6y=0$; find its intersections with the axes, and find and draw the tangent at each. Draw the line which will pass through the highest and lowest points of the ellipse, and the line which will pass through the extreme right and left points. Sketch the ellipse.

123. The Parabola.—Given the equation

$$y^2-2xy+x^2-4y+x-2=0, \tag{1}$$

for which $b^2-4ac=0$, to find its locus. Solving for y, we get

$$y=x+2\pm\sqrt{3(x+2)}. \tag{2}$$

As with the ellipse,

$$y=x+2 \tag{3}$$

is a diameter bisecting vertical chords.

If x is less than -2, y is imaginary, so that there are no points of the curve to the left of $x=-2$, which is the vertical limiting tangent. As x increases, $\sqrt{3(x+2)}$, the length of the half-chord, increases without limit, remaining real, so that the curve is open, spreading indefinitely.

If we put $x=1$, we find

$$y=1+2\pm\sqrt{3(1+2)}=3\pm3=6 \text{ or } 0.$$

The points $(1, 6)$ and $(1, 0)$ thus lie on the parabola. Any number of points may be located in this way, but we have enough to *trace* the curve, as in Fig. 23.

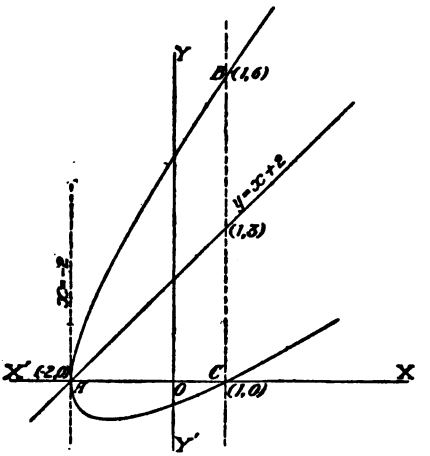


FIG. 23

124. Tracing a Parabola.—The steps necessary in tracing a parabola may be summarized:

Given an equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

for which $B^2-4AC=0$:

(1) Solve for y in terms of x.

(2) Trace the oblique diameter.

(3) Draw the vertical limiting tangent.

(4) Compute y for some one value of x, and plot the corresponding points.

(5) Sketch a parabola through these two points and tangent to the vertical limiting tangent where it crosses the oblique diameter.

125. Diameters of the Parabola.—The equation of a parabola can always be solved in a variety of ways; for instance, besides solving the equation of Art. 123 for y, we can solve for x in terms of y, getting

$$x=y-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{12y+9},$$

showing that the parabola is intersected by the diameter $y=x+\frac{1}{2}$ where this diameter is crossed by the horizontal limiting tangent $y=-\frac{3}{4}$.

Again, we can write immediately

$$y-x=\pm\sqrt{4y-x+2},$$

showing that the parabola is intersected by the diameter $y=x$ where this diameter is crossed by the oblique limiting tangent $4y-x+2=0$.

126. Construction of the Parabola.—It can be shown that any number of points may be located on a parabola as follows:

Given a parallelogram $KK'BC$, Fig. 24, bisected by a line which passes through A, the middle point of KK', and A', the middle point of CB. Lay off any lengths along AA' and number the points of division consecutively. Lay off the same

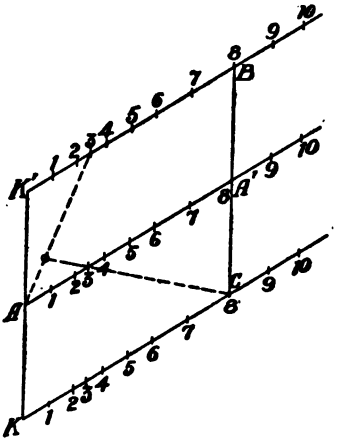


FIG. 24.

lengths, correspondingly numbered, along $K'B$ and KC. Join C with any point of AA', and A with the corresponding point of $K'B$; the two lines will meet at a point P of the parabola tangent to KK' at A, passing through B and C, and having AA' as the diameter that bisects chords parallel to KK'.

Obviously lines from C and A give only half the parabola; lines from B to points of AA' and from A to corresponding points of KC give the other half.

Examples.

1. Trace $x^2+2xy+y^2-x-4y+3=0$, finding also the slope where it crosses the y-axis.

2. Trace $4x^2+4xy+y^2+4y-8x-5=0$, finding the slopes where it crosses the axes.

3. At any point of the parabola of Example 2,

$$\frac{dy}{dx} = \frac{-2(2x+y-2)}{2x+y+2};$$

write the condition that this direction shall be perpendicular to that of the diameters. What is the slope of the line represented by this equation? At what point does this line cut the parabola? What is the line?

4. Find the locus of points equally distant from $y+x=1$ and the point $(1, 1)$, and trace the curve.

5. Given a point $P(2, 3)$ and two straight lines (1) $2y=x$ and (2) $y=3x$, find and trace the two loci (a) of points equidistant from P and (1); and (b) of points equidistant from P and (2). Thus locate the points equidistant from P, (1) and (2).

127. The Hyperbola.—In the equation whose second-degree terms may be resolved into two unequal factors, there are peculiar properties of its locus which require special consideration. Suppose the simplest form of such an equation be taken,

$$xy=1, \tag{1}$$

where the linear factors of the second-degree term are x and y. Solving for y, and then for x, we obtain

$$y=\frac{1}{x}, \tag{2}$$

$$x=\frac{1}{y}. \tag{3}$$

Since x and y can be interchanged without changing the equation, we infer that the curve is symmetrical to the line $y=x$ (the dotted line in the figure) and the point $(1, 1)$ is on the curve.

Beginning with this point, arrange the corresponding values of x and y in series:

$$x: \quad 1, 2, 3, 4, \ldots, 100, \ldots, \infty,$$

$$y: \quad 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \tfrac{1}{100}, \ldots, 0.$$

It is evident that as x increases indefinitely, y decreases indefinitely; i. e., the curve approaches indefinitely near the x-axis; or in symbols, when x is ∞, y is 0, and the curve touches the x-axis at infinity. It may be shown in the same way that the curve is tangent to (touches) the axis of x at $-\infty$, and from equation (3), that the y-axis is also tangent at $\pm\infty$.

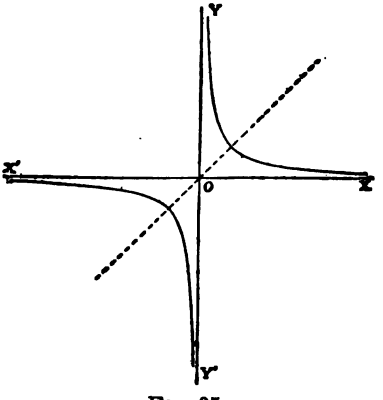


FIG. 25.

The curve is called a *hyperbola,* and the *two lines having the peculiar property of approaching indefinitely close to the curve as the curve becomes indefinitely distant from the origin are called asymptotes.*

If we find the intersections with the curve of any straight line through the origin, $y=mx$, we obtain the points

$$\left(\frac{1}{\sqrt{m}}, \sqrt{m}\right), \left(-\frac{1}{\sqrt{m}}, -\sqrt{m}\right);$$

that is, every chord through the origin is bisected at the origin, which is therefore the *center of the curve,* and *such a chord is a diameter.*

128. Asymptotes of the Hyperbola.—The locus of any equation

$$ax^2+bxy+cy^2+dx+ey+f=0$$

having $b^2-4ac>0$ (that is, any hyperbola) has two asymptotes.

For instance, suppose the equation is

$$x^2+xy-6y^2-3x+y-4=0,$$

having $b^2-4ac=25$.

Factoring the terms of highest degrees (this we may do with any such equation), we have

$$(x-2y)(x+3y)-3x+y-4=0. \qquad (1)$$

If we solve (1) simultaneously with some linear equation to get the intersections of the hyperbola and the corresponding straight line, we shall ordinarily find, on eliminating x, an equation of the second degree in y, which will determine two points of intersection. If the y equation happens to be of the first degree, we shall understand that one of the points has gone to infinity; and if y has dropped out of the equation altogether, it will mean that both intersections are at infinity; i. e., that the line is tangent to the hyperbola at infinity, or is an asymptote. (See Art. 46.)

If in (1) we substitute $x-2y=c$, we have

$$c(x+3y)-3x+y-4=0,$$

an equation of the *first* degree; it is evident, therefore, that the line $x-2y=c$ cuts the hyperbola *once* at infinity, whatever constant value c may have. Let us now determine c so that it shall cut twice at infinity; i. e., so that elimination of x between $x-2y=c$ and Equation (1) shall give an equation lacking both y^2 and y.

We are to determine c so that

$$[(x-2y)(x+3y)-3x+y-4]_{x=2y+c}=0$$

shall lack both y^2 and y terms; or so that

$$[c(x+3y)-3x+y-4]_{x=2y+c}=0$$

shall lack the y term.

Solving for c, then substituting for x and dividing both numerator and denominator of the second member by y, we have

$$c=\frac{5+\dfrac{3c+4}{y}}{5+\dfrac{c}{y}}.$$

In order that the line be tangent to the curve at infinity y must be infinite, which gives $c=1$. This value of c will cause y to vanish from the above equation, and hence

$$x-2y-1=0$$

will be an asymptote.

In the same way, we can show that the asymptote corresponding to the other factor is $x+3y=2$, as follows:

$$x+3y=c_1=\left[\frac{3x-y+4}{x-2y}\right]_{x=-3y+c_1}=2.$$

129. Asymptotes as the Form at Infinity.—A briefer explanation of the process of evaluating asymptotes is the following:

If the equations of the two asymptotes be multiplied together—thus $(x-2y-1)(x+3y-2)=0$—and compared with the equation of the curve, we have

$$\text{Asymptotes: } (x-2y)(x+3y)-3x+y+2=0,$$
$$\text{Curve: } (x-2y)(x+3y)-3x+y-4=0,$$

the two equations differing only in the constant term.

The asymptotes, the direction-ratio (m) of which is given by the factors of the highest-degree terms, determine the form of the curve at infinity; that is, when x and y become indefinitely great, the curve and the asymptotes coincide. But any line parallel to an asymptote, of the form $y=mx+c$, also meets the curve at infinity, since the ratio, $\frac{y}{x}$, is the same at infinity for all such lines.

This is shown by writing the line in the form

$$\frac{y}{x}=m+\frac{k}{x}\Big]_{y=mx=\infty}=m,$$

and evaluating the ratio $\frac{y}{x}$ for infinite values of x and y.

The particular line which has this direction-ratio, and at the same time coincides at infinity with the curve, will be derived by *evaluating the equation of the curve* for the given ratio at infinity,

$$y=mx=\infty.$$

This is done by writing the equation of the curve in the form

$$x-2y=\frac{3x-y+4}{x+3y}\Big]_{x=2y=\infty}=\frac{5+\frac{4}{y}}{5}\Big]_{y=\infty}=1.$$

$$x+3y=\frac{3x-y+4}{x-2y}\Big]_{x=-3y=\infty}=\frac{-10+\frac{4}{y}}{-5}\Big]_{y=\infty}=2.$$

130. Tracing a Hyperbola.—In order to *trace* a hyperbola, it is sufficient to have the asymptotes and one point of the curve. More points will improve the accuracy of the tracing, and it is necessary to observe the following, which can be shown to be properties of the hyperbola:

I. Any hyperbola is symmetrical with regard to the bisectors of the angle between its asymptotes; and

II. The intersection of the asymptotes is the center of the hyperbola, so that if P is a known point of one branch of the curve and A the intersection of the asymptotes, a point P' can be found on the other branch by producing the line PA until $AP'=PA$.

Ordinarily the most convenient points to find on the hyperbola are its intersections with the axes of coordinates; if the hyperbola meets neither axis, or if its intersections with the axes are inconvenient, a point can always be found by intersecting the hyperbola with a parallel to one of its asymptotes.

For instance, in

$$x^2+xy-6y^2-3x+y-4=0,$$

let $y=0$; then

$$x^2-3x-4=0, \quad x=4 \text{ or } -1;$$

let $x=0$; then

$$6y^2-y+4=0, \quad y \text{ is imaginary.}$$

After drawing the asymptotes and marking the computed points and the points obtained by carrying these across the center, we are ready to trace the curve as indicated in Fig. 26.

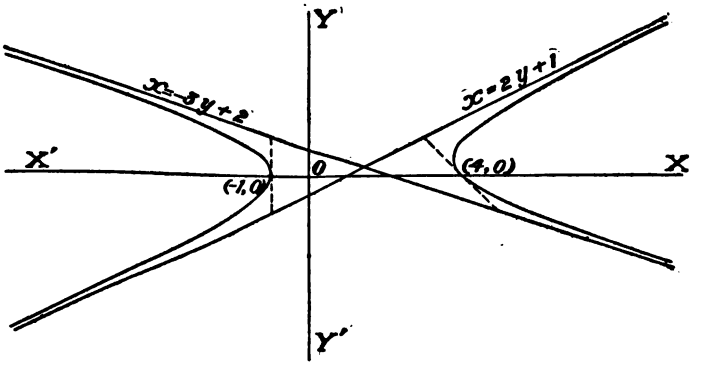


FIG. 26.

The tracing is also helped by finding the slope, $\frac{dy}{dx}$, at two or three points of the curve, and drawing the tangents. Here

$$\frac{dy}{dx} = \frac{-2x-y+3}{x-12y+1},$$

and is -1 at $(4, 0)$ and ∞ at $(-1, 0)$.

Examples.

Trace each of the following, finding its slope and drawing its tangent at each intersection with the axes:

1. $x^2-xy-2y^2+4y+x-6=0$.
2. $2x^2-5xy+2y^2-7x+8y=0$.
3. $3x^2-7xy+2y^2-2x+y-3=0$.

131. Construction of the Hyperbola.—If the asymptotes of a hyperbola are given, and one point P of the curve, any number of points on the hyperbola may be constructed as follows:

Draw any line through P meeting the asymptotes at K and L, and lay off on this line $LQ=KP$. Then Q is another point of the hyperbola.

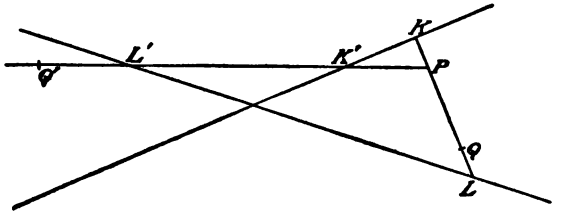


FIG. 27.

Fig. 27 shows two cases, one by primed letters. If we are given more than one point of the curve, each point may be used, and the construction thus checked.

132. Conjugate Hyperbolas.—The equation

$$(x-2y-1)(x+3y-2)=0$$

represents two intersecting straight lines, and

$$(x-2y-1)(x+3y-2)+c=0$$

represents, for any constant value of c, some hyperbola meeting these lines only at infinity; i. e., having them as asymptotes. If c is positive, the hyperbola will lie in one pair of vertical angles formed by the asymptotes, and if c is negative, it will lie in the other pair.

The two hyperbolas determined by

$$(x-2y-1)(x+3y-2)+a=0$$

and

$$(x-2y-1)(x+3y-2)-a=0$$

are said to be *conjugate* to each other. For instance, the hyperbola we have been discussing was

$$x^2+xy-6y^2-3x+y-4=0,$$

its asymptotes

$$x^2+xy-6y^2-3x+y+2=0,$$

so that the value of c for the original hyperbola was -6. For its conjugate, it is $+6$; the conjugate is therefore

$$x^2+xy-6y^2-3x+y+8=0.$$

Examples.

1. Trace $2y^2+xy-6x^2-10y-27x-24=0$ and its conjugate.
2. Trace $y^2+xy-2x^2-y+7x=0$ and its conjugate.
3. Find the equation of the locus of points $\frac{5}{4}$ as far from $(-2, 1)$ as from $y=2x$, and trace the curve.

133. An equation

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

if $b^2-4ac<0$, ordinarily represents an *ellipse;* if $b=0$ and $a=c$, it represents a *circle,* and it may happen that y is real for only one real value of x, or for none; in the first case the equation represents a single point, or *null ellipse,* in the second, an *imaginary ellipse.*

If $b^2-4ac=0$, the equation ordinarily represents a *parabola,* but it may happen to represent *two parallel* or *two coincident straight lines.*

If $b^2-4ac>0$, the equation ordinarily represents a *hyperbola,* but it may happen to represent *two intersecting straight lines.*

Miscellaneous Examples.

Trace the following:

1. $y^2-2xy+x^2-2y-x+4=0$.
2. $4x^2-12xy+13y^2+24x-52y+16=0$.
3. $x^2-xy=3$ and $y^2+xy=10$, showing intersections.
4. $x^2-y^2=0$ and $3x^2-4xy+5y^2=9$, showing intersections.
5. $5x^2+4xy-y^2+4y-8x-5=0$.
6. $4y^2-4xy+x^2+8y-12x-12=0$.
7. $5x^2+4y^2+4xy+4y-8x-5=0$.
8. $4x^2+49y^2-28xy+12x-42y+9=0$.
9. $4y^2-4xy+17x^2-8y+132x+196=0$.
10. $4x^2+49y^2-28xy+16x-56y+15=0$.
11. $2x^2+xy-6y^2-6x+16y-8=0$.
12. $4x^2-4xy+2y^2-8x+2y-4=0$.
13. $10x^2-51xy+5y^2+70x-60y+100=0$.
14. $16y^2-16xy+13x^2-32y-2x-11=0$.
15. $y^2-2xy+x^2-2x-2y+4=0$.
16. $5x^2+8xy+4y^2-21x-20y+19=0$.
17. $4x^2-16xy+16y^2+36x+24y-23=0$.
18. $y^2-xy-y+3x-6=0$.

19. $xy+2x-4y=0$.
20. $xy+x^2=3y$.
21. $xy+x^2-2y^2-3x=0$.
22. $4y^2+4xy+x^2-x-4=0$.
23. $4y^2-4xy+5x^2+8y-28x+40=0$.
24. $2y^2+7xy-4x^2-6x+3y-5=0$.
25. $3x^2-5xy-2y^2+6x-5y-3=0$.
26. $4y^2-4xy+65x^2+8y-68x-236=0$.
27. $2x^2-2xy+y^2+2x-4y+1=0$.
28. $y^2+xy-2x^2-4y-2x+8=0$.
29. $y^2-2xy+x^2+20y-21x+75=0$.

CHAPTER VIII.

Undetermined Coefficients

134. We have several times had occasion to determine a function when we knew its general form and one or more of its values. For instance, in finding the equation of a line through a given point parallel or perpendicular to a given straight line, we knew the general form of the equation to be $y=mx+b$, and from the given conditions we determined the values of the constants m and b. We used the same method in finding the tangent to a circle from an outside point, and in determining the asymptotes of a hyperbola. (Arts. 117 and 128.)

When a function is assumed with coefficients which are to be determined from given conditions, the coefficients are called *undetermined coefficients.*

In the example cited, m and b are undetermined coefficients in the function $mx+b$.

135. The method of undetermined coefficients is useful even when we do not know precisely the general form of the desired function, for we can apply it to any form that seems probable and then test the results for accuracy. For instance, suppose we think that $x^4+2x^3-35x^2+12x-1$ may have factors of the form $(x^2+ax-1)(x^2+bx+1)$. We assume the identity.

$$x^4+2x^3-35x^2+12x-1\equiv(x^2+ax-1)(x^2+bx+1),$$

which expresses the fact that its two members are different expressions for the same function and are therefore equal for any value of x. On this assumption, we have:

If $x=1$,

$$-21=a(2+b),$$

if $x=-1$,

$$-49=-a(2-b),$$

whence $a=7$, $b=-5$ and the factors are

$$(x^2+7x-1)(x^2-5x+1).$$

Multiplying these together, we find that the assumption was correct.

If we treat $x^4+2x^3-42x^2+12x-1$ in precisely the same way, we find $a=7$, $b=-6$, but the product of the factors

$$(x^2+7x-1)(x^2-6x+1)$$

is

$$x^4+x^3-42x^2+13x-1;$$

so that, in this case, the assumption is not correct.

Examples.

1. Find the equation of the line joining $(1, -3)$ and $(-2, 1)$ by assuming it to be $y=mx+b$, and determining m and b from the fact that each point lies on the line.

2. Find the tangent to $x^2+y^2=9$ through $(9, 7)$; first, assuming its equation to be $y=mx\pm 3\sqrt{1+m^2}$; second, assuming its equation to be $y=mx+b$ and determining m and b from the facts that $(9, 7)$ is on $y=mx+b$ and that the intersections of the circle and the line are coincident, so that $x^2+(mx+b)^2=9$ has equal roots, and thus a zero discriminant.

3. Factor $x^4-3x^3+7x-15$, assuming the factors to be of the form $(x^2+ax+3)(x^2+bx-5)$.

4. Factor $x^4+5x^3-4x^2-27x+21\equiv(x^2+ax-3)(x^2+bx-7)$.

136. Partial Fractions.—It is often necessary to transform a rational fraction into an equivalent sum of simpler rational fractions, called *partial fractions*. The following rules give assumptions that will lead to correct results:

The denominator of a given proper fraction having been factored as completely as possible:

(1) Corresponding to every linear factor $x-a$, assume a partial fraction of the form

$$\frac{A}{x-a}.$$

(2) Corresponding to any linear factor $x-b$ that occurs more than once, assume partial fractions of the form

$$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\cdots,$$

one for each repetition of the factor.

(3) Corresponding to a quadratic factor x^2+px+q, assume a partial fraction

$$\frac{Px+Q}{x^2+px+q}.$$

All the coefficients A, B, P and Q are constants.

Example 1: For example, suppose we wish to break up

$$\frac{x+13}{x^2+x-6}$$

into its partial fractions.

Assume the identity

$$\frac{x+13}{x^2+x-6}\equiv\frac{A}{x-2}+\frac{B}{x+3},$$

whence

$$x+13\equiv(x+3)A+(x-2)B.$$

The two members of this identity represent the same function, and so have the same values for any value of x. If we let $x=2$, and again let $x=-3$, we obtain:

If $x=2$,

$$15=5A,\quad A=3;$$

if $x=-3$,

$$10=-5B,\quad B=-2;$$

so that

$$\frac{x+13}{x^2+x-6}\equiv\frac{3}{x-2}-\frac{2}{x+3}.$$

Note that in order to determine one of the coefficients, we give x a value that will cause the other coefficient to drop out.

Example 2: As a second example, we will break up

$$\frac{x^2+2x+2}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}.$$

$$x^2+2x+2 \equiv A(x+1)^2 + B_1x(x+1) + B_2x.$$

Let $x=0$; then

$$2=A; \quad A=2.$$

Let $x=-1$; then

$$1=-B_2; \quad B_2=-1.$$

In order to find B_1, substitute the values of A and B_2 and simplify:

$$x^2+2x+2 \equiv 2(x+1)^2 + B_1x(x+1) - x,$$

$$-x^2-x \equiv B_1x(x+1).$$

As one form of the function is divisible by $x(x+1)$, the other must be; dividing the identity, we find

$$-1=B_1;$$

and we have

$$\frac{x^2+2x+2}{x(x+1)^2} \equiv \frac{2}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

Example 3: The following example illustrates the use of the imaginary $i=\sqrt{-1}$: Separate

$$\frac{1}{x^2(x^2+1)} \equiv \frac{B_1}{x} + \frac{B_2}{x^2} + \frac{Px+Q}{x^2+1}.$$

$$1 \equiv B_1x(x^2+1) + B_2(x^2+1) + (Px+Q)x^2.$$

Let $x=0$; then

$$1 \equiv B_2;$$

substitute and simplify, dividing out x; then

$$-x \equiv B_1(x^2+1) + (Px+Q)x.$$

Let $x=0$; then

$$0=B_1;$$

substitute, simplify, and divide out x; then

$$-1\equiv Px+Q.$$

Let $x=i$; then

$$-1=Pi+Q,$$

whence $P=0$, $Q=-1$, as the real and imaginary parts must be separately equal.

We have

$$\frac{1}{x^2(x\mp 1)}=\frac{1}{x^2}-\frac{1}{x^3+1}.$$

It must be observed that these methods apply only to fractions of which the numerator is of lower degree than the denominator; i. e., to proper fractions. An improper fraction must first be separated by division into an integral expression plus a proper fraction.

Examples.

Check your results by summing the partial fractions.

1. $\dfrac{10x-26}{x^3-4x^2+x+6}$.

2. $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$.

3. $\dfrac{x^2}{(x^2-1)^2}$.

4. $\dfrac{x}{x^4-x^2-2}$.

5. $\dfrac{x^2-x+2}{x^4-5x^2+4}$.

6. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$.

7. $\dfrac{x^2}{(x-1)^2(x^2+1)}$.

8. $\dfrac{3x^3-8x^2+10}{(x-3)(x^2+2x-5)}$.

9. $\dfrac{x^2+ax+a^2}{x^4-a^4}$.

10. $\dfrac{-2x^3+15x^2+15x-10}{(x^2+x-2)^2}$.

137. Theorem of Undetermined Coefficients.—The following principle is often useful:

If two polynomials are identical, the coefficient of any power of their variable is the same in both.

This may be seen as follows: Given

$$a-bx+cx^2+dx^3+\ldots\ldots\equiv A+Bx+Cx^2+Dx^3+\ldots\ldots.$$

Being expressions for the same function, and therefore equal for any value of x, these polynomials are equal when $x=0$; hence

$$a=A,$$

and

$$bx+cx^2+dx^3+\ldots\ldots\equiv Bx+Cx^2+Dx^3+\ldots\ldots.$$

Divide out x; then

$$b+cx+dx^2+\ldots\ldots\equiv B+Cx+Dx^2+\ldots\ldots.$$

Again let $x=0$; then

$$b=B,$$

and

$$cx+dx^2+\ldots\ldots\equiv Cx+Dx^2+\ldots\ldots,$$

or

$$c+dx+\ldots\ldots\equiv C+Dx+\ldots\ldots.$$

Proceeding in this way, we see the truth of the proposition above. Note that after dividing out x from any one of these identities, we still have an identity; for x is not identically zero; i. e., zero is not the only value x can have. (See Art. 36.)

138. Partial Fractions by Comparison of Coefficients.—This theorem can be applied to partial fractions as follows: Separate

$$\frac{1}{x^2(x^2+1)}\equiv\frac{B_1}{x}+\frac{B_2}{x^2}+\frac{Px+Q}{x^2+1}.$$

$$1 \equiv B_1 x(x^2+1) + B_2(x^2+1) + (Px+Q)x^2,$$

or

$$1 \equiv (B_1+P)x^3 + (B_2+Q)x^2 + B_1 x + B_2.$$

Comparing coefficients, $B_1+P=0$, $B_2+Q=0$, $B_1=0$, $B_2=1$, whence

$$B_1=0,\ B_2=1,\ P=0,\ Q=-1.$$

Examples.

Separate into partial fractions:

1. $\dfrac{x^2-1}{x^4+x^2+1}$. (Compare coefficients.)

2. $\dfrac{x^3+4x}{x^3+1}$. (Put $x=-1$ in your identity, simplify the identity, and compare coefficients.)

3. $\dfrac{5x^2-2x}{2(x^2-x+2)(x^2+x-2)}$.

4. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$.

CHAPTER IX.

SERIES.

139. Definitions.—*Terms following one another, each produced from the preceding by application of the same process, or in other words according to the same law, form a series.*

If there is no limit to the number of terms that may be written in accordance with the law, the series is called an *infinite series;* if only a limited number of terms is possible, it is called a *finite series.* A series may be *numerical,* each of its terms being a constant number; or *variable,* its terms involving some variable quantity.

Thus the terms of a geometric progression form a series, a finite or an infinite series according as we consider a limited or an unlimited number of terms. The expansion of $(1+x)^n$ by the Binomial Theorem is a finite series if n is a positive integer, an infinite series if n is fractional or negative.

140. Power-Series.—Among the most commonly used series are those whose terms are of the form ax^k, where a is a constant, x the variable, and k an integer, positive or negative. Such a series, when its terms are arranged in ascending or descending powers of the variable x, is called an *ascending* or *descending power-series in* x. A polynomial so arranged is a finite power-series.

Examples:

Infinite ascending power-series:

$$1+\frac{x}{1}+\frac{x^2}{\underline{2}}+\frac{x^3}{\underline{3}}+\frac{x^4}{\underline{4}}+\cdots\cdots$$

Infinite descending power-series:

$$4x^2+2x+1+\frac{1}{2x}+\frac{1}{4x^2}+\frac{1}{8x^3}+\cdots\cdots$$

141. Use of Power-Series.—Infinite power-series are used to express fractional and irrational algebraic functions and various sorts of transcendental functions, because they are simpler than the original; for under certain conditions a power-series may be handled as a polynomial, giving results that may be used with any required degree of accuracy.

142. Coefficients of Identical Power-Series.—The *Theorem of Undetermined Coefficients*, in its application to infinite as well as finite power-series, is stated:

If two power-series in x are identically equal, the coefficients of like powers of x in the two series are equal.

The proof is identical with that given in Art. 137 for finite power-series, or polynomials.

143. Summation of Finite Series.—Certain finite series can be summed by the method of undetermined coefficients; for instance, the sum of the squares of the first n integers can be found as follows:

Assume the identity

$$1^2+2^2+3^2+4^2+\ldots+n^2 \equiv a+bn+cn^2+dn^3+en^4+\ldots,$$

where $a, b, c, \ldots$ are numbers independent of n, whose values are to be determined. As this formula is to be true for any number of integers, write it for $(n+1)$ integers:

$$1^2+2^2+3^2+4^2+\ldots+n^2+(n+1)^2$$
$$\equiv a+b(n+1)+c(n+1)^2+d(n+1)^3+e(n+1)^4+\ldots.$$

Subtracting the first identity,

$$n^2+2n+1$$
$$\equiv b+c(2n+1)+d(3n^2+3n+1)+e(4n^3+\ldots.)+\ldots.$$

Comparing coefficients, we see that e and all subsequent coefficients must be zero, and that

$$3d=1, \quad 2c+3d=2, \quad b+c+d=1,$$

whence

$$b=\tfrac{1}{6}, \quad c=\tfrac{1}{2}, \quad d=\tfrac{1}{3}.$$

$$1^2+2^2+3^2+\ldots.+n^2 \equiv a+\tfrac{1}{6}(n+3n^2+2n^3).$$

Putting $n=0$ in the original identity or putting $n=1$ (or any other integer) in the last identity, we get $a=0$, so that

$$1^2+2^2+3^2+4^2+\ldots.+n^2 \equiv \frac{n}{6}(n+1)(2n+1).$$

Examples.

Find, by the method of undetermined coefficients, the sum of n terms of the following series (the dot is used for a multiplication sign):

1. $1\cdot 2+2\cdot 3+3\cdot 4+\ldots..$ to n terms.

Ans. $\frac{n}{3}(n+1)(n+2)$.

2. $1^2+3^2+5^2+\ldots.+(2n-1)^2$ (n terms).

Ans. $\frac{n}{3}(2n-1)(2n+1)$.

3. $1^3+2^3+3^3+\ldots..$ to n terms. *Ans.* $\left[\frac{n}{2}(n+1)\right]^2$.

4. $1^3+3^3+5^3+\ldots..$ to n terms. *Ans.* $n^2(2n^2-1)$.

5. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots..$ to n terms.

Ans. $\frac{n}{4}(n+1)(n+2)(n+3)$.

6. $1\cdot 3\cdot 5+3\cdot 5\cdot 7+5\cdot 7\cdot 9+\ldots..$ to n terms.

Ans. $n(n+2)(2n^2+4n-1)$.

7. $1\cdot 2+3\cdot 4+5\cdot 6+\ldots..$ to n terms.

Ans. $\frac{n}{3}(n+1)(4n-1)$.

8. $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots.$ to n terms.

Ans. $\frac{n}{12}(n+1)(n+2)(3n+5)$.

144. Expansion in Series by Undetermined Coefficients.— When we attempt to find the quotient $\frac{1+x}{1-x}$ or the root $\sqrt{1+x}$, either by the elementary processes or by using the Binomial Theorem, we develop an infinite series; this is called *expanding* the given function into an infinite series. The results are sometimes more easily obtained by the use of undetermined coefficients; the process is illustrated in the following examples:

Example 1: Expand $\frac{1+x}{1-x}$ to four terms.

Assume

$$\frac{1+x}{1-x} \equiv a+bx+cx^2+dx^3+\ldots\ldots$$

Then

$$1+x \equiv \begin{array}{l|l|l|l} a+b & x+c & x^2+d & x^3+\ldots\ldots \\ -a & -b & -c & \end{array}$$

Comparing coefficients,

$$a=1, \quad b-a=1, \qquad c-b=0, \quad d-c=0;$$
$$a=1, \quad b=a+1=2, \quad c=b=2, \quad d=c=2.$$

Hence

$$\frac{1+x}{1-x} \equiv 1+2x+2x^2+2x^3+\ldots\ldots$$

Example 2: Expand $\sqrt{1+x}$ to four terms.

Assume

$$\sqrt{1+x} \equiv a+bx+cx^2+dx^3+\ldots\ldots$$

Then

$$1+x \equiv \begin{array}{l|l|l|l} a^2 & x+ & b^2 & x^2 \\ +2ab & +\ 2ac & & +2ad \\ & & & +2bc \end{array} \Big| \; x^3$$

Comparing coefficients,

$$a^2=1, \qquad 2ab=1, \qquad 2ac+b^2=0, \qquad 2bc+2ad=0;$$

$$a=\pm 1, \; b=\frac{1}{2a}=\pm\frac{1}{2}, \; c=-\frac{b^2}{2a}=\mp\frac{1}{8}, \; d=-\frac{bc}{a}=\pm\frac{1}{16}.$$

Hence
$$\sqrt{1+x} \equiv \pm\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots.\right).$$

Examples.

Expand by the method of undetermined coefficients:

1. $\frac{(1-x)}{(1+x)}$ to four terms, and check by division.

Ans. $1-2x+2x^2-2x^3+\ldots..$

2. $\sqrt[3]{1-x}$ to four terms, and check by the Binomial Theorem.

Ans. $1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}-\ldots..$

3. $\frac{1-8x}{1-x-6x^2}$ to four terms, and check first by division, next by separation into partial fractions and expansion of each fraction by the Binomial Theorem. *Ans.* $1-7x-x^2-43x^3-\ldots..$

145. Reversion of Series.—When an identity $y \equiv f(x)$ is solved so that x is expressed as a function of y, the function $f(x)$ is said to be *reverted*. The reversion, when $f(x)$ is a series, is performed by undetermined coefficients as in the following examples:

Example 1: Revert the series
$$y \equiv x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots.$$
to four terms.

Assume
$$x \equiv ay+by^2+cy^3+dy^4+\ldots.;$$
then
$$\begin{array}{r|l|l|l}
x \equiv ax-\frac{a}{2} & x^2+\frac{a}{3} & x^3-\frac{a}{4} & x^4+\ldots. \\
+\ b & -\ b & +\frac{b}{4} & \\
 & +\ c & +\frac{2b}{3} & \\
 & & -\frac{3c}{2} & \\
 & & +\ d &
\end{array}$$

whence

$$x \equiv y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \ldots\ldots$$

Example 2: Revert the series

$$y \equiv 1 + x + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \frac{x^4}{\underline{4}} + \ldots\ldots$$

An attempt to develop this in the form

$$x \equiv a + by + cy^2 + dy^3 + \ldots.$$

will fail. But assume

$$x \equiv a(y-1) + b(y-1)^2 + c(y-1)^3 + d(y-1)^4 + \ldots.;$$

then

$$\begin{array}{rl|l|l|l} x \equiv ax + & \frac{a}{2} & x^2 + & \frac{a}{6} & x^3 + \frac{a}{24} \\ + b & & + b & & + \frac{b}{4} \\ & & + c & & + \frac{b}{3} \\ & & & & + \frac{3c}{2} \\ & & & & + d \end{array} \left| x^4 + \ldots. \right.$$

whence

$$x \equiv (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \ldots\ldots$$

Examples.

Revert the series:

1. $y \equiv x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.$ to three terms.

Ans. $x \equiv y + \frac{y^3}{3} + \frac{2y^5}{15} + \ldots\ldots$

2. $y \equiv x - \frac{x^3}{\underline{3}} + \frac{x^5}{\underline{5}} - \ldots.$ to three terms.

Ans. $x \equiv y + \frac{y^3}{6} + \frac{3y^5}{40} + \ldots\ldots$

3. $y \equiv 1 - \frac{3x}{2} + \frac{15x^2}{8} - \frac{35x^3}{16} + \ldots$. to three terms.

Ans. $x \equiv -\frac{2}{3}(y-1) + \frac{5}{9}(y-1)^2 - \frac{40}{81}(y-1)^3 + \ldots\ldots$

4. $y \equiv 1 + x + x^2 + x^3 + x^4 + \ldots$. to four terms.

Ans. $x \equiv (y-1) - (y-1)^2 + (y-1)^3 - (y-1)^4 + \ldots\ldots$

5. $y \equiv ax + x^3$ to five terms.

Ans. $x \equiv \frac{y}{a}\left[1 - \frac{y^2}{a^3} + 3\frac{y^4}{a^6} - 12\frac{y^6}{a^9} + 55\frac{y^8}{a^{12}} - \ldots\ldots\right]$.

6. Given $x^3 + px + q = 0$, from the results of Example 5 write x in terms of p and q.

Ans. $x = -\frac{q}{p}\left[1 - \frac{q^2}{p^3} + 3\left(\frac{q^2}{p^3}\right)^2 - \ldots\ldots\right]$.

7. Apply the result of Example 6 to find a root of

$$x^3 - 6.75x + 1.75 = 0.$$

Ans. $x = \frac{7}{27}\left[1 + \left(\frac{196}{27^3}\right) + 3\left(\frac{196}{27^3}\right)^2 + \ldots\right] = 0.2619212.$

146. Difference Method of Series.—If in a given series we subtract each term from the succeeding one, there results a second series, the terms of which are called *first differences.*

A third series may be formed in the same way from the second, its terms forming the *second differences.*

If the process results finally in a series each term of which is very small, or zero, the original series is said to be *differenced.*

Example: The cubes of the natural numbers and the common logarithms to five places of the numbers 60, 61, 62, etc., form the following series of differences:

$f(x)$	D_1	D_2	D_3	D_4
1	7	12	6	0
8	19	18	6	0
27	37	24	6	
64	61	30		
125	91			
216				

No.	log.	D_1	D_2	D_3
60	1.77815	0.00718	−0.00012	+0.00001
61	1.78533	706	11	0
62	1.79239	695	11	0
63	1.79934	684	11	+0.00001
64	1.80618	673	−0.00010	
65	1.81291	0.00653		
66	1.81954			

In general, if the series is T_1, T_2, T_3, T_4, ...:

	D_1	D_2	D_3	D_4
T_1	$-T_1+T_2$	$T_1-2T_2+T_3$	$-T_1+3T_2-3T_3+T_4$	$T_1-4T_2+6T_3-4T_4+T_5$
T_2	$-T_2+T_3$	$T_2-2T_3+T_4$	$-T_2+3T_3-3T_4+T_5$	$T_2-4T_3+6T_4-4T_5+T_6$
T_3	$-T_3+T_4$	$T_3-2T_4+T_5$	$-T_3+3T_4-3T_5+T_6$	
T_4	$-T_4+T_5$	$T_4-2T_5+T_6$		
T_5	$-T_5+T_6$			
T_6				

Using the D above each series of differences to denote the first term of that series, we have:

$D_1=-T_1+T_2.$	$T_1=T_1.$
$D_2=T_1-2T_2+T_3.$	$T_2=T_1+D_1.$
$D_3=-T_1+3T_2-3T_3+T_4.$	$T_3=T_1+2D_1+D_2.$
$D_4=T_1-4T_2+6T_3-4T_4+T_5.$	$T_4=T_1+3D_1+3D_2+D_3.$
$D_5=-T_1+5T_2-10T_3+10T_4-5T_5+T_6.$	$T_5=T_1+4D_1+6D_2+4D_3+D_4.$
	$T_6=T_1+5D_1+10D_2+10D_3+5D_4+D_5.$

The numerical coefficients in these expressions for T_1, T_2, T_3, T_4, T_5 are the coefficients in the expansions of $(1+x)^0$, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$. It can be proved that:

In general, the $(n+1)$th term of the series is

$$T_{n+1}=T_1+nD_1+\frac{n(n-1)}{\lfloor 2}D_2+\frac{n(n-1)(n-2)}{\lfloor 3}D_3 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}D_4+\ldots.,$$

where the coefficients are those of the expansion of $(1+x)^n$.

147. Interpolation.—This expression for the nth term after the first in the series is applicable whether n is a whole number or a fraction, the meaning of the $\frac{1}{2}$ or $\frac{2}{3}$ term after the first being the term $\frac{1}{2}$ the way or $\frac{2}{3}$ the way between the first and second, in the sense in which we used these expressions in describing the process of interpolation in handling a table of logarithms. We spoke at that time of the method there described as only approximately correct; the formula just written is the exact interpolation formula. When the second differences are zeros, as is nearly true in most parts of a logarithmic table, the nth term after the first is given as T_1+nD_1, the ordinary rule for interpolation. For accurate interpolation, more terms of the formula above are needed as long as they affect the last decimal place desired.

Example: log 62.3, if we call log 62 the first term, is the $1\frac{3}{10}$ term, and $T_1=1.79239$, $D_1=0.00695$, $D_2=-0.00011$;

$$\log 62.3 = 1.79239+\tfrac{3}{10}\times 0.00695+\frac{\frac{3}{10}\left(-\frac{7}{10}\right)}{2}\times(-0.00011)$$

$$= 1.79239$$
$$+0.002085$$
$$+0.000012$$
$$\log 62.3 = 1.79449,$$

as may be verified by the tables.

148. The formula of Art. 146, besides being useful for interpolation, serves sometimes to determine the law of a series of which some terms are known, without the law being at once obvious. For this purpose the formula for the nth term is commonly more useful than that for the $(n+1)$th. It is

$$T_n=T_1+(n-1)D_1+\frac{(n-1)(n-2)}{1\cdot 2}D_2$$
$$+\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}D_3+\ldots.$$

For instance, suppose the numbers of our first example of Art. 146 were given and we wished to find their law of sequence. We should have:

$$T_n = 1 + (n-1)7 + \frac{(n-1)(n-2)}{1\cdot 2}12 + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}6 + \dots.$$

$$= \begin{array}{r|l|l|l} 1 & +7 & n+6 & n^2+n^3=n^3; \\ -7 & -18 & -6 & \\ +12 & +11 & & \\ -6 & & & \end{array}$$

i. e., each term is the cube of the number of the term.

149. Summation of Series.—It is evident that the $(n+1)$th term of the series

$$(a): 0,\ T_1,\ T_1+T_2,\ T_1+T_2+T_3,\ T_1+T_2+T_3+T_4,\ \dots.$$

is the sum of the first n terms of the series

$$(b): T_1,\ T_2,\ T_3,\ T_4,\ \dots.,$$

and also that the series of first differences of series (a) is identical with series (b), so that any series of differences of (a) is the series of differences one earlier for (b). Hence the sum of n terms of series (b) is T_{n+1} of the series (a); or

$$S_n = nT_1 + \frac{n(n-1)}{\lfloor 2}D_1 + \frac{n(n-1)(n-2)}{\lfloor 3}D_2 + \dots.$$

For instance, the sum of the cubes of the first n positive integers (Art. 146) is

$$S_n = n + \frac{n(n-1)}{2}7 + \frac{n(n-1)(n-2)}{6}12 + \frac{n(n-1)(n-2)(n-3)}{24}6$$

$$= \frac{n}{4}\{4 + 14(n-1) + 8(n-1)(n-2) + (n-1)(n-2)(n-3)\}$$

$$= \frac{n^2}{4}(n+1)^2.$$

Examples.

Find by the difference method the nth term and the sum of n terms of each of the following:

1. 2, 6, 12, 20, 30, 42, ...

Ans. $T_n=n(n+1)$; $S_n=\frac{n}{3}(n+1)(n+2)$.

2. 4, 12, 28, 52, 84, 124, ...

Ans. $T_n=4(n^2-n+1)$; $S_n=\frac{4n}{3}(n^2+2)$.

3. 2, 9, 28, 65, 126, 217, ...

Ans. $T_n=n^3+1$; $S_n=\frac{n}{4}(n^3+2n^2+n+4)$.

4. 6, 24, 60, 120, 210, 336, ...

Ans. $T_n=n(n+1)(n+2)$;

$S_n=\frac{n}{4}(n+1)(n+2)(n+3)$.

5. 4, 18, 48, 100, 180, 294, ...

Ans. $T_n=n(n+1)^2$; $S_n=\frac{n}{12}(n+1)(n+2)(3n+5)$.

6. 2, 9, 20, 35, 54, ...

Ans. $T_n=(n+1)(2n-1)$; $S_n=\frac{n}{6}(4n^2+9n-1)$.

7. -1, 0, 15, 56, 135, 264, 455, ...

Ans. $T_n=n(2n-1)(n-2)$; $S_n=\frac{n}{6}(n+1)(3n^2-7n+1)$.

8. Find the term midway between the first and second in Example 6 and Example 7.

9. Find the sum of the squares of the first n integers, and the number half-way between the first and second terms.

10. Given $f(x)\equiv x^3-5x^2+4x-3$, show that the fourth order of differences of the series $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ is zero. Find by interpolation $f(2.9)$ and $f(0.5)$.

11. Given the series 0.5176, 0.6840, 0.8452, 1.0000, 1.1472, 1.2856, the lengths of the chords subtending 30, 40, 50, 60, 70 and 80° at the center of a circle of radius 1.0000; difference the series, and by interpolation find to four decimals the chords corresponding to 35° and 46°. *Ans.* 0.6014 and 0.7814.

150. Sum of an Infinite Series—Convergence and Divergence.—In the case of the infinite geometric series,

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,$$

we have seen that if the common ratio r is less than unity in numerical value, the sum of n terms approaches a definite limit as n is indefinitely increased, or, in formulæ,

$$S_n = \frac{a(1-r^n)}{1-r}\Big]_{n=\infty} = \frac{a}{1-r}.$$

151. Definitions.—When S_n, the sum of n terms of a series, approaches a definite limit S as the number of terms, n, is indefinitely increased, the value S is called the *sum of the series.* The series is said to be *convergent,* and to *converge* toward the number S.

Any series which is not thus convergent is called a *divergent* series. For example, a geometric progression in which r is numerically greater than unity is divergent because $S_n = a\,\frac{r^n-1}{r-1}$ can be made greater than any assignable quantity by making n large enough.

152. Convergence Unaffected by Omitted Terms.—It is evident that if the first few terms of an infinite series be removed, the infinite series left will converge or diverge according as the original series converges or diverges, and vice versa.

153. Absolutely Convergent Series.—*A convergent series which contains only positive terms, or a convergent series which remains convergent after the signs of all its negative terms are changed, is called an absolutely convergent series, and is said to converge absolutely.*

For instance, 10, -5, $\frac{5}{2}$, $-\frac{5}{4}$, $\frac{5}{8}$, $-\frac{5}{16}$, etc., is a geometric progression of which the common ratio, $-\frac{1}{2}$, is >-1 and <1, and is therefore convergent. Moreover, since the series 10, 5, $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$, ... is also convergent, the first series converges absolutely.

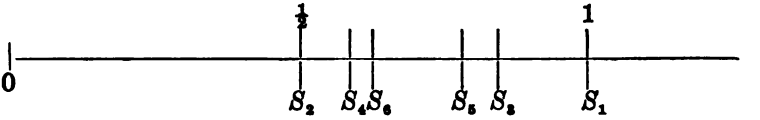
154. Alternating Series.—*If the terms of an infinite series eventually alternate in sign, and become smaller and smaller the further the series progresses, the n-th term approaching zero as a limit as n is indefinitely increased, the series is convergent.*

This proposition is almost self-evident; the general proof is the same as for the following example:

Consider the series 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$, $\frac{1}{5}$, $-\frac{1}{6}$, ...

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots . \pm \frac{1}{n}.$$

Plot points to represent the successive values of S_n:



The odd S's move toward the left, the even toward the right; the difference between the inner two at any time is equal to the numerical value of the term last taken (i. e., $= \frac{1}{n}$ when n terms have been taken), and so approaches zero as n is indefinitely increased. The intermediate value upon which the S's are closing in is the sum, S, of the series. If the terms did not alternate at the start, the discussion would be the same, the sum of the terms before the alternating begins taking the place of 0 in the figure.

A similar study of the series 2, $-1\frac{1}{2}$, $1\frac{1}{4}$, $-1\frac{1}{8}$, ... shows that, though its terms alternate in sign and decrease constantly, it is an oscillating divergent series, because its nth term does not approach zero as a limit as $n = \infty$.

The series 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$, $\frac{1}{5}$, $-\frac{1}{6}$, ... is not absolutely convergent, because the series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... diverges, as appears if we group its terms as follows:

$$1 + \tfrac{1}{2} > \tfrac{1}{2};$$
$$\tfrac{1}{3} + \tfrac{1}{4} > 2(\tfrac{1}{4}), \text{ or } \tfrac{1}{2};$$
$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > 4(\tfrac{1}{8}), \text{ or } \tfrac{1}{2};$$
$$\tfrac{1}{9} + \tfrac{1}{10} + \ldots . + \tfrac{1}{15} + \tfrac{1}{16} > 8(\tfrac{1}{16}), \text{ or } \tfrac{1}{2}; \text{ and so on.}$$

Out of the terms of this series, we can take as many groups of terms as we please, each group amounting to more than $\frac{1}{2}$; consequently, we can make S_n as large as we please by taking n large enough.

155. Region and Limits of Convergence.—We have seen what is meant by the convergence or divergence of a numerical series; when the variable of a variable series is given a particular constant value, the resulting numerical series may, in most cases, be either convergent or divergent, according to the value of the constant so substituted. For instance, we know that the geometric series $3x+3x^2+3x^3+\ldots$. will furnish a convergent series if any positive or negative proper fraction be put in place of x, but a divergent series if any other number be substituted for x.

The values of its variable for which a variable series becomes convergent form its region of convergence; the boundaries of this region are called its limits of convergence.

A series may or may not converge for its limits of convergence; in the case of the geometric series, the limits of convergence are -1 and $+1$, for which values the series diverges.

156. Use of Convergent Series.—*The sum of a variable infinite series has a definite value for every value of the variable within the region of convergence;* and so, within that region, *expresses a function of the variable.* Outside of that region, the series has no sum and no meaning, and is useless.

The sum of the first few terms of an infinite series is a polynomial; it can be shown that if the series is absolutely convergent, the results obtained by using this polynomial are approximately the same as if the function defined by the whole series had been used. The approximation can be made as close as desired by taking enough terms. In this way, when a function is expanded into a series, its value can be computed for any given value of its variable; expressions can also be formed for the sum, product, quotient, etc., of any functions expressed in series.

As series are simpler and easier to handle than any other expression for some functions, and are the only means of expressing others, they are very important. Most of the practical work of mathematics is performed with the aid of series.

A series that converges, but not absolutely, is of little use; its sum can be changed by merely rearranging its terms; moreover, such a series converges very slowly.

Example.

Expand $\frac{1+2x}{1+x}$ (by actual division) into a series in two ways, getting $\frac{1+2x}{1+x} \equiv 1+x-x^2+x^3-x^4+\ldots.$ and $\frac{2x+1}{x+1} \equiv 2-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}\ldots..$

Show that the region of convergence of the first series is between -1 and $+1$, and the region of convergence of the second series is outside of this. Compare the values of S_4 of the first series with the values of $\frac{1+2x}{1+x}$ when $x=\frac{1}{2}$, 0.1, 0.01, respectively, and the values of S_4 of the second series with the values of $\frac{1+2x}{1+x}$ when $x=2$, 10, 100, respectively.

157. Ratio of Convergence.—The notation introduced in the study of the progressions is used for all series; namely, T_k for the kth term, T_1, T_2, T_3, ... for the first, second, third, ... terms,

$$S_n = T_1+T_2+T_3+\ldots.+T_n,$$

$$S_n]_{n=\infty} = S_\infty \text{ or simply } S.$$

The ratio of any term, T_{n+1}, to the next preceding, T_n, is called the *ratio of convergence,* or the *test-ratio* of the series; it is represented by r_n.

$$r_n = \frac{T_{n+1}}{T_n}.$$

158. Tests for Convergence.—From what we have already seen, an obvious *necessary* test for convergence is that the value of an individual term shall approach zero as a limit as we go indefinitely further out in the series; i. e., that

$$[T_n]_{n=\infty} = 0.$$

Unless, in addition, the signs of the terms of the series alternate, this is not *sufficient* for convergence, and even then does not ensure absolute convergence.

Consider the test-ratio, or ratio of convergence,

$$r_n = \frac{T_{n+1}}{T_n}.$$

This commonly varies as we take terms at different places, and we shall show that if, as we go further and further out in the series, it approaches a limit numerically less than 1, the series will converge absolutely; if it approaches a limit numerically greater than 1, or increases without limit, the series will diverge; if it approaches 1 or -1 as a limit, the series may either converge or diverge.

159. Comparison-Test.—First, if we have two series,

$$t_1,\ t_2,\ t_3,\ t_4,\ \ldots.$$

and

$$T_1, T_2, T_3, T_4, \ldots.,$$

the second of which is known to be convergent, and if each term of the first series is numerically less than the corresponding term of the second series, then it is evident that the first series is convergent.

For instance, since

$$1+\tfrac{1}{2}+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3+(\tfrac{1}{2})^4+\ldots.$$

is convergent, it is evident that

$$1+\tfrac{1}{2}(\tfrac{1}{2})+\tfrac{1}{3}(\tfrac{1}{2})^2+\tfrac{1}{4}(\tfrac{1}{2})^3+\tfrac{1}{5}(\tfrac{1}{2})^4+\ldots.$$

is also convergent.

The test by means of the ratio of convergence is derived by comparing any given series with a geometric progression.

Given any series

$$T_1, T_2, T_3, T_4, \ldots, \tag{1}$$

and supposing that from the kth term on, the ratio of convergence is numerically less than some constant number a which itself is numerically less than 1 (i. e., $r_n < a < 1$), then the series (1) is convergent, for

$$\frac{T_{k+1}}{T_k} < a, \quad \text{or} \quad T_{k+1} < aT_k;$$

$$\frac{T_{k+2}}{T_{k+1}} < a, \quad \text{or} \quad T_{k+2} < aT_{k+1} < a^2T_k;$$

and so on.

Each term of the series

$$T_k, T_{k+1}, T_{k+2}, T_{k+3}, \ldots. \tag{2}$$

is numerically less than the corresponding term of the series

$$aT_k, a^2T_k, a^3T_k, a^4T_k, \ldots. \tag{3}$$

The series (3) is a geometric progression with common ratio a numerically less than 1, and so is convergent; hence the series (2) is convergent, and also the series (1), from which the series (2) was obtained by the omission of a definite number of terms.

The second part of the test can be proved by using in a similar way the evident principle that if every term of a given series is numerically greater than the corresponding term of a divergent series, the given series is also divergent.

When $[r_n]_{n=\infty} = \pm 1$, neither of these comparisons will apply; there are in fact both convergent and divergent series for which r_n is limit is numerically equal to unity.

160. Test by the Limit of the Ratio of Convergence.—This test is formulated for reference as follows:

$\left[r_n = \frac{T_{n+1}}{T_n}\right]_{n=\infty} = L$, and L numerically <1, series converges.

$\left[r_n = \frac{T_{n+1}}{T_n}\right]_{n=\infty} = L'$, and L' numerically >1, series diverges.

$\left[r_n = \frac{T_{n+1}}{T_n}\right]_{n=\infty} = \pm 1$, no test.

Examples.

Test the following series for convergence, giving the region of convergence:

1. $1+nx+n(n-1)\frac{x^2}{\underline{2}}+n(n-1)(n-2)\frac{x^3}{\underline{3}}+\ldots.$

$$+\frac{\underline{n}}{\underline{n-k}}\frac{x^k}{\underline{k}}+\ldots.. \quad \textit{Ans.}\ -1<x<1.$$

2. $1+x+\frac{x^2}{\underline{2}}+\frac{x^3}{\underline{3}}+\frac{x^4}{\underline{4}}+\ldots.. \quad \textit{Ans.}\ -\infty<x<+\infty.$

3. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.. \quad \textit{Ans.}\ -1<x<1.$

4. $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots.. \quad \textit{Ans.}\ -1<x<1.$

5. $\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\ldots..$

Ans. $x>0$ or <-1.

6. $x-\frac{x^3}{\underline{3}}+\frac{x^5}{\underline{5}}-\frac{x^7}{\underline{7}}+\ldots.. \quad \textit{Ans.}\ -\infty<x<+\infty.$

7. $1-\frac{x^2}{\underline{2}}+\frac{x^4}{\underline{4}}-\frac{x^6}{\underline{6}}+\ldots.. \quad \textit{Ans.}\ -\infty<x<+\infty.$

8. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots.. \quad \textit{Ans.}\ -1<x<1.$

9. $\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\frac{1}{7x^7}+\ldots..$

Ans. $x<-1$ or $>+1$.

10. $1+cx+\frac{c^2x^2}{\underline{2}}+\frac{c^3x^3}{\underline{3}}+\frac{c^4x^4}{\underline{4}}+\ldots.. \quad \textit{Ans.}\ -\infty<x<\infty.$

161. The Exponential Theorem.—The exponential function, a^x, in which x is the variable and a is any constant, is expanded into an ascending power-series as follows:

Assume

$$a^x \equiv b + cx + dx^2 + ex^3 + \ldots\ldots \quad (1)$$

As this is an identity, x can be replaced by any number; write $2x$ for x:

$$a^{2x} \equiv b + 2cx + 4dx^2 + 8ex^3 + \ldots\ldots \quad (2)$$

But also $(a^x)^2 \equiv a^{2x}$:

$$\begin{aligned} a^{2x} \equiv\ & b^2 + c^2x^2 + d^2x^4 + e^2x^6 + \ldots\ldots \\ & + 2bcx + 2bdx^3 + 2bex^5 + 2bfx^7 + 2bgx^9 + 2bhx^{11} + \ldots\ldots \\ & + 2cdx^3 + 2cex^5 + 2cfx^7 + 2cgx^9 + \ldots\ldots \\ & + 2dex^7 + 2dfx^9 + \ldots\ldots \end{aligned} \quad (3)$$

and so on.

Comparing coefficients in (2) and (3):

$$\begin{aligned} b^2 &= b. & b &= 1. \\ 2bc &= 2c. & c &= c. \\ 2bd + c^2 &= 4d. & d &= \frac{c^2}{2} = \frac{c^2}{\underline{|2}}. \\ 2cd + 2be &= 8e. & e &= \frac{c^3}{6} = \frac{c^3}{\underline{|3}}. \\ 2ce + 2bf + d^2 &= 16f. & 14f &= \frac{c^4}{3} + \frac{c^4}{4}; & f &= \frac{c^4}{24} = \frac{c^4}{\underline{|4}}. \\ 2de + 2cf + 2bg &= 32g. & 30g &= \frac{c^5}{6} + \frac{c^5}{12}; & g &= \frac{c^5}{120} = \frac{c^5}{\underline{|5}}. \\ 2df + 2cg + 2bh + e^2 &= 64h. & 62h &= \frac{c^6}{24} + \frac{c^6}{60} + \frac{c^6}{36}; & h &= \frac{c^6}{720} = \frac{c^6}{\underline{|6}}. \\ \ldots\ldots & & \ldots\ldots & & & \end{aligned}$$

We have so far been able to determine the desired coefficients only in terms of one of their own number, c, which is itself still undetermined.

We have

$$a^x \equiv 1 + cx + \frac{c^2}{\underline{|2}}x^2 + \frac{c^3}{\underline{|3}}x^3 + \frac{c^4}{\underline{|4}}x^4 + \ldots + \frac{c^n}{\underline{|n}}x^n + \ldots\ldots$$

This series converges for any value of x; and if we put $x=\frac{1}{c}$, we find

$$a^{\frac{1}{c}}=1+1+\frac{1}{\underline{2}}+\frac{1}{\underline{3}}+\frac{1}{\underline{4}}+\ldots\ldots$$

The value of this numerical series is represented by e; the computation of e to five decimals is as follows:

$$\begin{aligned}
&2.000000\\
\frac{1}{\underline{2}}&= .500000;\ \div 3=\\
\frac{1}{\underline{3}}&= .166667;\ \div 4=\\
\frac{1}{\underline{4}}&= .041667;\ \div 5=\\
\frac{1}{\underline{5}}&= .008333;\ \div 6=\\
\frac{1}{\underline{6}}&= .001389;\ \div 7=\\
\frac{1}{\underline{7}}&= .000198;\ \div 8=\\
\frac{1}{\underline{8}}&= .000025;\ \div 9=\\
\frac{1}{\underline{9}}&= .000003\\
e&=2.71828
\end{aligned}$$

Now since $a^{\frac{1}{c}}=e$,

$$e^c=a,$$
$$c=\log_e a,$$

and the value of c is definitely determined.

The use of the variable as an exponent has introduced a new function, of the utmost importance in mechanics and physics, the treatment of which is impossible without the use of infinite series.

We have finally,

$$a^x \equiv 1 + (x \log_e a) + \frac{(x \log_e a)^2}{\lfloor 2} + \frac{(x \log_e a)^3}{\lfloor 3} + \frac{(x \log_e a)^4}{\lfloor 4} + \ldots,$$

where

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots = 2.71828, \text{ approx.}$$

In particular, since $\log_e e = 1$,

$$e^x \equiv 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots.$$

162. The Logarithmic Series.—Expanding $(1+x)^y$ by both the Binomial Theorem and the Exponential Theorem, we have

$$(1+x)^y \equiv 1 + yx + \frac{y(y-1)}{\lfloor 2} x^2 + \frac{y(y-1)(y-2)}{\lfloor 3} x^3 + \frac{y(y-1)(y-2)(y-3)}{\lfloor 4} x^4 + \ldots,$$

$$(1+x)^y \equiv 1 + y \log_e(1+x) + \frac{y^2}{\lfloor 2} [\log_e(1+x)]^2 + \frac{y^3}{\lfloor 3} [\log_e(1+x)]^4 + \ldots.$$

These series are identical, so the coefficient of y is the same in both: in the second, it is $\log_e(1+x)$; in the first, it is

$$x + \frac{(-1)}{\lfloor 2} x^2 + \frac{(-1)(-2)}{\lfloor 3} x^3 + \frac{(-1)(-2)(-3)}{\lfloor 4} x^4 + \ldots.$$

Hence

$$\log_e(1+x) \equiv x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots. \qquad (1)$$

This series is convergent if x is numerically less than 1, and so will give the logarithm to the base e (the *natural* or *hyperbolic* logarithm) of any number from 0 to 2. It converges slowly, except for values of x near 0.

Replacing x by $-x$, we find

$$\log_e(1-x) \equiv -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots . \qquad (2)$$

and since $\log \frac{1+x}{1-x} \equiv \log(1+x) - \log(1-x)$,

$$\log_e \frac{1+x}{1-x} \equiv 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots .\right]. \qquad (3)$$

This series converges for any value of x between -1 and 1, and so will give the natural logarithm of any positive number, but its convergence is slow unless x is small.

COMPUTATION OF NATURAL LOGARITHMS.

For the computation of natural logarithms, we may put $\frac{1+x}{1-x} = \frac{n+1}{n}$, which gives, when solved, $x = \frac{1}{2n+1}$ in terms of n; then

$$\log_e \frac{n+1}{n} \equiv 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots .\right],$$

or

$$\log_e(n+1) \equiv \log_e n + 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \ldots .\right]. \qquad (4)$$

This series converges for any positive value of n.

We know that $\log_e 1 = 0$; we can therefore find the logarithm of any positive integer as indicated below:

$$\log_e 2 = (\log_e 1 = 0) + 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \ldots .\right],$$

$$\log_e 3 = \log_e 2 + 2\left[\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \ldots .\right],$$

$$\log_e 4 = \log_e 3 + 2\left[\frac{1}{7} + \frac{1}{3 \cdot 7^3} + \frac{1}{5 \cdot 7^5} + \frac{1}{7 \cdot 7^7} + \ldots .\right] = 2\log_e 2,$$

and so on.

This computation is checked whenever the logarithm of a composite number is found both by series and by adding the logarithms of its factors. The computation of $\log_e 2$ is given to show the method:

$\frac{1}{3}$	0.33333333 : 1	0.33333333
$(\frac{1}{3})^3$	0.03703704 : 3	1234568
$(\frac{1}{3})^5$	411523 : 5	82305
$(\frac{1}{3})^7$	45725 : 7	6532
$(\frac{1}{3})^9$	5081 : 9	565
$(\frac{1}{3})^{11}$	565 : 11	51
$(\frac{1}{3})^{13}$	63 : 13	5
		0.34657359

$\log_e 2 = .6931472$

In a similar way, we find $\log_e 5 = 1.6094379$

Adding these, we find $\log_e 10 = 2.3025851$

163. Computation of Common or Briggs Logarithms. Modulus.—We have

$$\log_a x \equiv \frac{\log_e x}{\log_e a};$$

that is, the logarithm of any number to the base a is found by multiplying the natural logarithm of the number by $\frac{1}{\log_e a} = \log_a e$. This multiplier is called the *modulus* of the system of logarithms having the base a.

In particular, the modulus of the common system, usually denoted by μ, is

$$\mu = \frac{1}{\log_e 10} = \frac{1}{2.3025851} = 0.434294+.$$

$$\log_{10} x \equiv \mu \log_e x \equiv 0.434294 \log_e x, \text{ approx.}$$

For instance,

$$\log_{10} 2 = 0.434294 \times 0.693147 = 0.301030.$$

Examples.

1. Compute $\log_e 3$ to 7 places by putting $n=2$ in series (4) and again by putting $n=8$.
2. Compute $\log_e 5$ to 7 places.
3. Compute $\log_e 10$ to 7 places, using $\log_e 3$.
4. Compute $\log_e 7$ to 7 places.
5. Compute $\log_e 11$ to 7 places.
6. Find the value of $\frac{1}{e}$ from the series for e^x.
7. Find the value of e^2 from the series for e^x, and check the work by $(2.71828+)^2$.
8. Expand $\left(1+\frac{1}{n}\right)^n$ to six terms, and show that

$$\left(1+\frac{1}{n}\right)^n\Big]_{n=\infty} = e.$$

9. Expand $\left(1+\frac{1}{n}\right)^{nx}_{n=\infty} = \left[\left(1+\frac{1}{n}\right)^n\right]^x_{n=\infty}$, and thus derive e^x.
10. Express $\frac{1}{2}(e^{ix}+e^{-ix})$ as an ascending power-series. ($i=\sqrt{-1}$.)
11. Express $\frac{i}{2}(e^{ix}-e^{-ix})$ as an ascending power-series.
12. If $y \equiv \log_e(1+x)$, show that $x \equiv e^y-1$; then by reverting the series $x \equiv y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\frac{y^4}{\lfloor 4}+\ldots$. find four terms of $\log_e(1+x)$.

CHAPTER X.

Choice and Chance.

164. Choice.—We have seen (Art. 49) that out of n different things, s may be chosen at a time in ${}_nP_s$ different permutations, or ${}_nC_s$ different combinations, where

$${}_nP_s = n(n-1)(n-2)\ldots(n-s+1) = \frac{\lfloor n}{\lfloor n-s},$$

$${}_nP_n = \lfloor n,$$

$${}_nC_s = \frac{\lfloor n}{\lfloor s \, \lfloor n-s} = \frac{n(n-1)(n-2)\ldots(n-s+1)}{1\cdot 2\cdot 3\cdot \ldots \cdot s} = {}_nC_{n-s},$$

$${}_nC_n = 1.$$

165. Permutations of n Things, of Which p are Alike of One Kind and q are Alike of Another Kind.—Suppose all such permutations to be made, x in number, and in each one a distinguishing mark to be put on each of the similar things, so as to make all the n things different; thus, if there are p of the kind a, and q of the kind b, like things would be $a_1a_2\ldots a_p$, $b_1b_2\ldots b_q$. Then from any one of the x permutations we could make $\lfloor p$ different arrangements without altering the position of any of the remaining letters; and hence we could make $x\lfloor p$ permutations out of the original x.

Similarly of the q letters b, when marked, we could make $x\lfloor p \cdot \lfloor q$ permutations out of the preceding $x\lfloor p$.

With the marks, the n letters are now all different, and admit of $\lfloor n$ permutations, hence

$$x\lfloor p \, \lfloor q = \lfloor n,$$

$$x = \frac{\lfloor n}{\lfloor p \, \lfloor q}.$$

The same method would lead, if extended to three kinds of letters, to the formula

$$ {}_nP_{p+q+r=n} = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}. $$

Examples.

1. How many different arrangements can be made out of the letters in $a^3b^2c^4$, written out at full length? *Ans.* 1260.

2. How many arrangements can be made with 15 poker-chips, 4 being red, 5 blue, and 6 white? *Ans.* 630630.

3. With 3 black, 3 red and 2 white flags, how many signals can be made by putting all the flags up at once? *Ans.* 560.

4. In how many ways can 10 men be divided into squads of (a) 3 and 7; (b) 1, 2 and 7; (c) 1, 2, 3, and 4?

Ans. (a) 120; (b) 360; (c) 12600.

5. In how many ways can 10 men be divided into a squad of 5 for duty in the trenches and a squad of 5 for guard duty? In how many ways into two indifferent squads of 5 each?

Ans. 252 and 126.

6. In how many ways can a pack of 52 playing-cards be divided equally among four players? In how many ways into 4 piles, the arrangement of the piles being indifferent?

Ans. $\frac{\lfloor 52}{(\lfloor 13)^4}$ and $\frac{\lfloor 52}{4(\lfloor 13)^4}$.

7. In how many ways can a king, a queen and a knave be chosen from a pack of playing-cards? *Ans.* 64.

166. Probability or Chance.—If one of $(m+n)$ events must occur, and only one can occur, and if any one of the $(m+n)$ is as likely to occur as any other, the *probability* or *chance* that one of the m events shall occur is defined to be the number $\frac{m}{m+n}$.

In order to be consistent, we must then define the chance that one of the n events shall occur as $\frac{n}{m+n}$, and the chance that one of the $(m+n)$ events shall occur as $\frac{m+n}{m+n}$, or 1. As by

hypothesis one of the $(m+n)$ events *must* occur, the definition gives unity as the chance of a certainty; any other chance is a proper fraction—except the chance of an impossibility, which is zero, being the limiting case of an indefinitely diminishing chance.

Example: If 3 black balls and 2 red balls are shaken up in a bag, and one is drawn at random, the chance that a black ball be drawn is $\frac{3}{5}$, and the chance that a red ball be drawn is $\frac{2}{5}$.

If the occurrence of any one of the m events above insures the occurrence of an event E, and the occurrence of any one of the n events prevents the occurrence of E, and if no other event has any influence on the occurrence of E, the chance $\frac{m}{m+n}$ is the chance of success for E, and the chance $\frac{n}{m+n}$ is the chance of failure for E.

The sum of the chances of success and of failure for any event is unity.

The value $\frac{m}{n}$ is called the *odds in favor* of the event E; $\frac{n}{m}$ the *odds against* the event E.

In the fall of 2 dice, 36 events are possible; for, calling the dice A and B, any one of the 6 faces of A may come up with any one of the 6 faces of B. The chance of success in an attempt to make a total of 8 in a single throw is therefore $\frac{5}{36}$; because each of the 5 events $A6$, $B2$; $A5$, $B3$; $A4$, $B4$; $A3$, $B5$; $A2$, $B6$ insures success, and any one of the other 31 insures failure. The chance of failure is $\frac{31}{36}$. The chance of the dice falling somehow is 1; the chance of their not falling at all is 0. The odds for the event are 5 to 31; against the event, 31 to 5.

Events are *independent* if the success or failure of one has no effect on the success or failure of any other. For example, in a throw of two dice, the event of any particular number coming up on one of the dice is independent of the event of any particular number coming up on the other die.

Events are *mutually exclusive* if only one of them can happen; that is, if the success of any one necessitates the failure of every other. For example, in a throw of a single die, the coming up of one number excludes the coming up of any other; any two or more of the counts 1, 2, 3, 4, 5, 6 in a throw of a single die are mutually exclusive events.

Events neither independent nor mutually exclusive are *dependent.* Success in drawing at random for an ace in a pack of cards is affected by every card drawn out of the pack (and not returned); that is, the success of any drawing is dependent upon the preceding drawings. The chance of success in an attempt to draw at random 1 of the 4 aces from a pack of 52 cards is, at the first drawing, $\frac{4}{52}$; if the card drawn is not an ace, the chance of success in the second drawing is greater; namely, $\frac{4}{51}$; if the card taken in the first drawing is an ace, the chance of success in the second drawing is less, namely, $\frac{3}{51}$.

167. Method of Computing a Chance.—In order to determine the chance of success of any event directly, it is necessary:

(1) To count all the events which affect the happening of the given event, finding that there are, say, N of them.

(2) To count all the events that insure the success of the given event, finding, say, M of them.

(3) To make sure that all the events thus considered are equally probable.

(4) To divide M by N, $\frac{M}{N}$ being the required chance.

In many cases, the necessary counting is best done by application of the principles of permutations and combinations.

For instance, the 52 cards of a pack are thoroughly shuffled; what is the chance that the 4 aces are together? $\lfloor\underline{52}$ arrangements of the cards are possible. $\lfloor\underline{4}\,\lfloor\underline{49}$ of these arrangements have the 4 aces together; any one arrangement is as likely to occur as any other, and the arrangement of the cards is all that

affects the happening of the required event; the chance is thus

$$\frac{\lfloor 4 \lfloor 49}{\lfloor 52} = \frac{24}{52 \cdot 51 \cdot 50} = \frac{1}{13 \cdot 17 \cdot 25} = \frac{1}{5525}.$$

The odds are 5524 to 1 against the event.

Again, a company of 20 men is selected by lot from a company of 25 men; what is the chance that a particular man be chosen? ${}_{25}C_{20}$ different selections may be made; if the particular man be omitted from the choice, ${}_{24}C_{20}$ may be made. The chance of his *not* being taken is thus

$$\frac{{}_{24}C_{20}}{{}_{25}C_{20}}$$

or

$$\frac{\lfloor 24}{\lfloor 20 \lfloor 4} \div \frac{\lfloor 25}{\lfloor 20 \lfloor 5} = \frac{5}{25} = \frac{1}{5}.$$

The chance of his being taken is $\frac{4}{5}$.

Examples.

1. From a bag containing 3 black balls and 2 red balls, 2 balls are drawn at random. What is the chance that they are of the same color? *Ans.* $\frac{2}{5}$.

2. Find independently the chance that the 2 balls of Example 1 are of different colors. What is the check on the result?

3. From 5 black, 3 red and 2 white balls, 3 are drawn at random. What is the chance that 1 is red and 2 are black? *Ans.* $\frac{1}{4}$.

4. If 10 persons form a row, what is the chance that a certain couple stand together? If they form a ring? *Ans.* $\frac{1}{5}$ and $\frac{2}{9}$.

5. From a bag containing 5 white, 3 red, and 4 black balls, 3 are drawn at random. Find the odds against all three being of one color. *Ans.* 41 to 3.

6. Three coins are tossed; what is the chance that they all fall heads up? That 2 do? That 1 does? That at least 2 do? That at least 1 does? *Ans.* $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{7}{8}$.

7. Two dice are thrown; show that the chances of the totals 2, 3, 4, ..., 10, 11, 12 are respectively $\frac{1}{36}$, $\frac{2}{36}$, $\frac{3}{36}$, ..., $\frac{6}{36}$, ..., $\frac{3}{36}$, $\frac{2}{36}$, $\frac{1}{36}$.

8. Three dice are thrown; show that the chance of three sixes is $\frac{1}{216}$; of triplets, $\frac{1}{36}$; of doublets, $\frac{5}{12}$; of a total of 5, $\frac{1}{36}$; of a total of 10 or 11, $\frac{1}{8}$.

9. Thirteen cards are chosen at random from a pack of 52; what is the chance that the 4 aces are among them? That they are all hearts? *Ans.* $\frac{11}{4165}$ and $\frac{\lfloor 13 \lfloor 39}{\lfloor 52}$.

168. Chance Determined by Observation.—In all of the foregoing discussion, the chance for success of an event appears merely as a mathematical idea, arbitrarily defined; it has also, however, been found by experiment that in a large number of trials the number of successes divided by the total number of events will give very closely the mathematical chance.

For instance, if 1 of 25 cards is marked, and 20 are drawn from them at random, it is found that, in 10,000 drawings, the marked card will appear in about 8000 drawings and fail to appear in the other 2000. Conversely, if, among 10,000 mill-operatives of a certain class, an average of 90 a year are killed while at work, the chance that a particular operative of this class will be killed in the course of any year while at work is assumed to be $\frac{90}{10000}$ or .009.

169. Chance of a Compound Event.—It often happens that the success of an event may be seen to depend on the success of other events of which the chance is known by observation or by experiment, or may be computed more readily than the chance of success of the first event. In this connection, the following principles are used:

1. *If the chance that a certain event shall happen is C_1, if the chance, after it has happened, that a second event shall happen is C_2, if the chance after both have happened that a third event shall happen is C_3, and so on, then the chance that all the events shall happen is the product, $C_1C_2C_3\ldots$, of the respective chances that each shall happen.*

If the events are independent, the stipulation "after the preceding events have happened" has no effect; if the events are dependent, it is essential.

2. *If, of a number of mutually exclusive events, the chances of success are respectively* C_1, C_2, C_3, ..., *the chance that one or another of them will happen is* $C_1+C_2+C_3+\ldots$., *the sum of the respective chances.*

To see the reason for the first of these principles, consider this example: In every thousand persons living under certain conditions, an average of 4 in every year have typhoid fever; of every hundred cases of typhoid an average of 5 are fatal; what is the chance that any one person of average constitution living under these conditions will die of typhoid in any one year? For convenience, consider 100,000 of these persons; .004 of them (400) will have typhoid, and of these .05 (20) will die; or, more simply, $.05\times.004$ of all will die of typhoid. Then the chance of death by typhoid will be $\frac{20}{100000}$ or $.05\times.004=.0002$. This is obviously merely the chance (.004) that the person shall have typhoid multiplied by the chance (.05) that, having it, he shall die of it.

A very simple, but perfectly general, exemplification of the second principle is the following: Two dice, *A* and *B*, are thrown; what is the chance that the total count be either 7 or 11? Of the 36 possible throws, 6, namely *A*6, *B*1; *A*5, *B*2; *A*4, *B*3; *A*3, *B*4; *A*2, *B*5; *A*1, *B*6, give a total of 7; and 2, namely *A*6, *B*5; *A*5, *B*6, give a total of 11. It clearly makes no difference whether we divide the total number of successful events (6+2) by 36, in order to find the chance of success, or add the respective chances ($\frac{6}{36}$ and $\frac{2}{36}$) of throwing 7 and of throwing 11, for

$$\frac{6+2}{36}=\frac{6}{36}+\frac{2}{36}.$$

Example.

1. Three cards are chosen at random from a pack of 52; show that the chance of getting an ace, a deuce and a trey is $\frac{1}{52}\times\frac{4}{51}\times\frac{4}{50}$.

2. Four cards are drawn at random from a pack of 52; find the chance that they are of different suits. *Ans.* $\frac{2197}{20825}$.

3. 120 men are to be formed into a solid rectangle, 10 men along one side and 12 along the other; all sides of the rectangle are equally likely to be in front. What is the chance that any one man will be in front? *Ans.* $\frac{11}{120}$.

4. *A*, *B* and *C* draw lots for a prize; *A* is to draw with *B*, the winner with *C*, and the winner of the second drawing is to take the prize. What are the chances? *Ans.* *A* $\frac{1}{4}$; *B* $\frac{1}{4}$; *C* $\frac{1}{2}$.

5. Supposing that every year $\frac{3}{5}$ of the candidates for admission to the Naval Academy enter, $\frac{5}{6}$ of the fourth class, $\frac{12}{13}$ of the third class, and $\frac{26}{27}$ of the second class are promoted, $\frac{8}{9}$ of the first class graduate, and 81 per cent of the Passed Midshipmen are commissioned; what is the chance of a candidate's eventually being commissioned? *Ans.* (Disregarding the possibility of trying twice at any stage) 32 per cent.

170. *Example 1:* The following example shows the use of these principles: Each of 3 men, *A*, *B*, and *C*, is in turn to toss 3 coins; the round is to be repeated until one tosses 2 heads and a tail and wins a prize. What is each man's chance of winning the prize?

The throws of course are independent events, and as each of the 3 coins may fall in either of 2 ways, there are 2^3 different events, of which 3 (according as one or another of the three coins falls tails) are 2 heads and a tail.

The chance of success at any one throw is then $\frac{3}{8}$ of the chance that the throw is made.

The chance of success at the first throw is $\frac{3}{8}$; of failure, $\frac{5}{8}$; the chance that the first throw fails and the second succeeds is $\frac{5}{8}\times\frac{3}{8}$; the chance that both fail is $\frac{5}{8}\times\frac{5}{8}$; the chance that

the first two throws fail and the third succeeds is $(\frac{5}{8})^2 \times \frac{3}{8}$; the chance that the first three fail is $(\frac{5}{8})^3$; and so on.

The 1st, 4th, 7th, ... throws belong to *A*.
The 2d, 5th, 8th, ... throws belong to *B*.
The 3d, 6th, 9th, ... throws belong to *C*.

The throws belonging to one man are exclusive events; hence one man's chance of success is the sum of the chances of success at his various possible throws:

For *A*: $\frac{3}{8}[1+(\frac{5}{8})^3+(\frac{5}{8})^6+(\frac{5}{8})^9+\ldots.]$.
For *B*: $\frac{5}{8} \times \frac{3}{8}[1+(\frac{5}{8})^3+(\frac{5}{8})^6+(\frac{5}{8})^9+\ldots.]$.
For *C*: $(\frac{5}{8})^2 \times \frac{3}{8}[1+(\frac{5}{8})^3+(\frac{5}{8})^6+(\frac{5}{8})^9+\ldots.]$.

The chance that even the seventh throw be made (i. e., that the first 6 all fail to give 2 heads and a tail) is very small; it is $(\frac{5}{8})^6$, or less than 0.06; still, it is barely conceivable that a successful throw will never occur. Hence, to get the total chance of each man, we must sum to infinity the geometrical progression, for which

$$S_\infty = \frac{1}{1-(\frac{5}{8})^3} = \frac{512}{512-125} = \frac{512}{387}.$$

The chances of success for *A*, *B* and *C* are thus:

$$\frac{3}{8} \times \frac{512}{387} = \frac{64}{129}, \quad \frac{5}{8} \times \frac{64}{129} = \frac{40}{129}, \quad \frac{5}{8} \times \frac{40}{129} = \frac{25}{129}.$$

The events of success for *A*, *B*, *C*, respectively, are exclusive; so that, with an infinite number of rounds, we have, as the chance of success of some one of the men,

$$\frac{64}{129} + \frac{40}{129} + \frac{25}{129} = 1.$$

If the prize in this case is a pool of \$1.29, it is fair for the first thrower to contribute \$0.64, the second \$0.40, the last \$0.25.

171. *Example 2:* The following example is somewhat similar to the preceding, but involves *dependent events.* The aces and kings are taken from a pack of cards and shuffled. Each of two persons, A and B, is to draw a card and keep it; the drawings are to be repeated in the same order until one person gets a heart. What is each person's chance of drawing a heart?

As there are 2 hearts among the 8 cards, the chance that A will succeed in the first drawing is $\frac{2}{8}$; that he will fail, $\frac{6}{8}$. In case he fails, there are 7 cards left to draw from, with 2 hearts among them; so the chance that B will succeed is $\frac{2}{7}$ of his chance of making the draw, or $\frac{2}{7}\times\frac{6}{8}$. The chance that B makes his draw and fails is $\frac{5}{7}\times\frac{6}{8}$, and is the chance that A makes a second draw. The chance that A succeeds on his second draw is $\frac{2}{6}\times(\frac{5}{7}\times\frac{6}{8})$; that he fails, $\frac{4}{6}\times(\frac{5}{7}\times\frac{6}{8})$. So we may go on, the chance that the seventh draw will occur being

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}=\frac{1}{28};$$

if A gets a chance to make this draw, he is sure to win, as there will be only 2 cards left, both hearts.

The chances of success on the seven drawings are mutually exclusive, so that the chances of success for A and B are:

$$\text{For } A: \quad \frac{1}{4}+\frac{5}{28}+\frac{3}{38}+\frac{1}{28}=\frac{4}{7}.$$

$$\text{For } B: \quad \frac{3}{14}+\frac{1}{7}+\frac{1}{14}\qquad =\frac{3}{7}.$$

172. *Example 3:* The following example is another illustration of the chance of a series or combination of events.

A marksman who makes 80 per cent of hits with a perfect rifle is shooting with one which rates 75 per cent of hits under perfect conditions. What is his chance of hitting (a) at any one shot, (b) 3 times in 3 shots, (c) just 3 times in 5 shots, (d) at least 3 times in 5 shots?

The answer to (a) is evidently $\frac{4}{5}\times\frac{3}{4}=\frac{3}{5}$; and to (b), $(\frac{3}{5})^3=\frac{27}{125}$.

In (c), he must make 3 hits, the chance of each of which is $\frac{3}{5}$, and 2 misses, the chance of each of which is $\frac{2}{5}$. The chance of a single combination of 3 hits and 2 misses is thus $(\frac{3}{5})^3(\frac{2}{5})^2$; but there are ${}_5C_3=10$ such combinations, so the chance is

$$10(\tfrac{3}{5})^3(\tfrac{2}{5})^2=\tfrac{216}{625}=.3456.$$

In (d), he must hit just 3, 4, or 5 times; the sum of the chances of these events is

$${}_5C_3(\tfrac{3}{5})^3(\tfrac{2}{5})^2+{}_5C_4(\tfrac{3}{5})^4(\tfrac{2}{5})+(\tfrac{3}{5})^5$$
$$=.3456+.2592+.07776=.68256.$$

The result for (d) can also be found by first computing his chance of failure; that is, the chance of 3, 4, or 5 misses, which is

$${}_5C_3(\tfrac{2}{5})^3(\tfrac{3}{5})^2+{}_5C_4(\tfrac{2}{5})^4(\tfrac{3}{5})+(\tfrac{2}{5})^5$$
$$=.2304+.0768+.01024=.31744.$$

The chance of success is

$$1-.31744=.68256.$$

(NOTE.—Instead of ${}_5C_3$ combinations of 3 hits and 2 misses, we can count the *orders* in which the 5 letters $h\ h\ h\ m\ m$ can be arranged.)

Examples.

1. A bag contains 6 black balls and 1 red. A person is to receive a dollar for every ball he draws at random until he draws the red one. What is his chance worth? *Ans.* \$3.

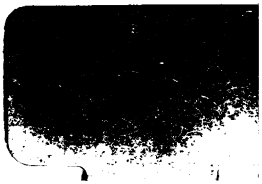
2. Of 2 chess players, 1 wins on the average 2 games out of 3. What is the chance (drawn games being disregarded) that he will win just 4 games out of 6? *Ans.* $\frac{80}{243}$.

3. A bag contains 4 black balls and 1 red ball. Every time a ball is drawn it is replaced and the bag shaken. What is the chance that in 5 drawings the red ball be drawn (a) at least once, (b) just once, (c) at least twice, (d) just twice?

Ans. (a) $\frac{2101}{3125}$, (b) $\frac{256}{625}$, (c) $\frac{821}{3125}$, (d) $\frac{128}{625}$.

4. A pool of 10 dollars is made up by *A*, *B*, *C* and *D*. *A*, *B* and *C* have each in turn 1 chance at tossing 3 coins; any one of them who tosses 2 heads and a tail thereby wins the pool, which otherwise goes to *D*. What is the value of each man's chance?

Ans. *A*'s, \$3.75; *B*'s, \$2.34; *C*'s, \$1.46; *D*'s, \$2.44.

5. A gun-crew makes an average of 2 per cent of hits; what is the chance of its making 2 hits in 100 shots? (Use logarithms.) *Ans.* 0.596.

6. A midshipman gets on the average 4 problems correct out of 10. On an examination paper of 8 questions, each of which is to be marked zero or perfect, what is his chance of getting a satisfactory mark? *Ans.* Between 0.173 and 0.174.